8th GRACM International Congress on Computational Mechanics,
Volos, 12 July - 15 July 2015

RADIATION AND PRESSURE GRADIENT EFFECTS ON THE INCOMPRESSIBLE LAMINAR BOUNDARY LAYER FLOW

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Keywords: approximate semi-analytical solutions, radiation effects, pressure gradient, boundary layer.

Abstract: *In fluid mechanics and aerospace engineering radiation and pressure gradient effects have a significant role in flow separation and heat transfer. In this study, we utilize classical perturbation theory and semi-analytical techniques for ordinary differential equations (ODEs) to obtain semi-analytical solutions of the problem under consideration. The partial differential equations (PDEs) of the problem are the continuity, the Navier-Stokes and the energy equations. We assume incompressible and laminar flow past a flat plate with pressure gradient and radiation parameter. Using the dimensionless Falkner-Skan transformation we obtain a non-linear and coupled system of two PDEs, that has a parabolic nature. By using a perturbation method, keeping terms of order up to* ε^2*, the system of the two PDEs is transformed into a system of six ODEs. In this study, we use two semi-analytical techniques, the Homotopy Analysis Method (HAM) and the Differential Transformation Method (DTM) to solve the ODE system. The results are compared with the numerical solution of the ODE system. Adverse pressure gradient decreases the dimensionless velocity of the boundary layer. Favorable pressure gradient have the opposite effect for the dimensionless velocity profile. The combination of adverse pressure gradient and radiation increase the temperature of the boundary layer in the case of a cooling plate and decrease the temperature in the case of a heating plate.*

1 INTRODUCTION

The laminar, incompressible boundary layer flow is a common and challenging problem. It has been studied for years but still several aspects of the problem remain unexplored. Blasius was the first to describe the steady two dimensional boundary layer flow that forms on a semi-infinite flat plate which is held parallel to a constant unidirectional flow [1]. Since then, many attempts have been made to solve the Blasius problem, both analytically and numerically. Howarth, numerically examined the adverse pressure gradient on the laminar boundary-layer flow [2]. Minkowycz and Sparrow found non-similar solutions for natural convection flow on a vertical cylinder utilizing perturbation techniques [3]. The last decades many attempts have been made to solve problems like these analytically. Ariel et al. studied the axisymmetric flow over a stretching sheet utilizing the homotopy perturbation method [4]. The variational iteration method was utilized by He to find approximate analytical solution of the Blasius problem [5]. Adomian proposed a methodology to solve analytically nonlinear problems, with which the nonlinear problem is deformed to a set of linear problems, called the Adomian decomposition method [8]. A breakthrough for analytical solutions not only for Blasius' problem but for many nonlinear equations was Liao's homotopy analysis method (HAM) [6, 7]. Using homotopy, Liao transforms a nonlinear problem into a finite number of linear problems. Recently, the differential transformation method (DTM), originally proposed by Zhu, has been utilized for the Blasius and other nonlinear problems [9, 10].

There has been extensive research on the effects of adverse pressure gradient and radiation on fluid flow [11, 17]. Cebeci and Bradshaw [12] have extensively studied the physical and computational aspects of convective heat transfer. England and Emery studied the thermal radiation effects on the laminar free convection boundary layer [13]. The effects of radiation in an optically thin gray gas flowing past a vertical infinite plate in the presence of a magnetic field has been previously studied by Raptis et al. [15]. Radiation effects on flow past a stretching plate have been studied by Xenos [14].

In this study, we obtain the non-dimensional form of the governing equations of the problem (continuity, momentum and energy equations) utilizing the Falkner-Skan transformation [16]. The obtained non-dimensional system of PDEs is transformed into an ODE system using perturbation methods [3]. This ODE system is solved in a semi-analytical way utilizing a hybrid method, combining HAM and DTM. We compare these semi-analytical results with the obtained numerical solution of the system of ODEs. The effects of pressure gradient (adverse or favorable) and thermal radiation are examined and analyzed. The developed methodology is compared with a numerical solution and achieves a very good precision, as shown in figures and tables.

2 MATHEMATICAL FORMULATION AND FALKNER-SKAN TRANSFORMATION

The governing equations that describe the two-dimensional laminar flow of an incompressible fluid over a flat plate are the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

the momentum equation in x-direction after the boundary-layer simplifications:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

and the energy equation:

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = u\frac{dp}{dx} + k\frac{\partial^2 T}{\partial y^2} - \mu\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial q_r}{\partial y}. \tag{3}$$

The boundary conditions are defined as follows:

$$\begin{gathered} y = 0: \quad u = 0, \quad v = 0, \quad T = Tw, \\ y \to \delta: \quad u = u_e, \quad T = T_e, \end{gathered} \tag{4}$$

where T_w is the temperature of the flat plate, T_e is the temperature at the edge of the boundary layer and

$$-\frac{\partial q_r}{\partial y} = 4\alpha\sigma(T_e^4 - T^4), \tag{5}$$

is the local radiant absorption, where α is the absorption coefficient and σ is the Stefan-Boltzmann constant [17]. We assume that the temperature differences within the flow are sufficiently small such that T^4 may be expressed as a linear function of the temperature. This is accomplished by expanding T^4 in a Taylor series about T_e and neglecting higher-order terms [14, 15], thus $T^4 \approx 4T_e^3 T - 3T_e^4$, and equation (5) takes the form:

$$-\frac{\partial q_r}{\partial y} = 16\alpha\sigma T_e^3(T_e - T). \tag{6}$$

Introducing the dimensionless *Falkner-Skan* transformation and the dimensionless stream function [12]:

$$\eta = y\sqrt{\frac{u_e}{\nu x}}, \qquad \Psi = \sqrt{u_e \nu x} f(x,\eta), \tag{7}$$

continuity equation is satisfied identically. At the edge of the boundary layer, we consider the linearly retarded flow, known as Howarth's flow, in which the external velocity varies linearly with x and the velocity at the edge is $u_e = V_\infty(1-x)$, where V_∞ is the free stream velocity. The flow at the edge of the boundary layer can be associated with the pressure drop using the Bernoulli equation:

$$\frac{dP}{dx} = -\rho u_e\frac{du_e}{dx}. \tag{8}$$

The dimensionless temperature is defined as $\theta = \dfrac{T_w - T}{T_w - T_e}$. Finally, the momentum and energy equations, (2) and (3) become:

$$f''' + \frac{1}{2}ff'' + \frac{x}{u_e}\frac{du_e}{dx}\left[1 - (f')^2\right] = x\left[f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right], \tag{9}$$

$$\frac{1}{Pr}\theta'' + \frac{1}{2}f\theta' + \frac{16x\alpha\sigma T_e^3}{\rho C_p u_e}(1-\theta) = x\left(f'\frac{\partial\theta}{\partial x} - \frac{\partial f}{\partial x}\theta'\right) + \frac{xf'}{C_p(T_w - T_e)}u_e\frac{du_e}{dx} - \frac{u_e^2}{C_p(T_w - T_e)}(f'')^2, \tag{10}$$

where $(\cdot)'$ denotes $\partial(\cdot)/\partial\eta$, and $f = f(x,\eta)$ and $\theta = \theta(x,\eta)$, with boundary conditions:

$$\begin{aligned} \eta = 0: \quad & f(x,0) = f'(x,0) = 0, \quad \theta(x,0) = 0, \\ \eta \to \delta: \quad & f'(x,\eta) = 1, \quad \theta(x,\eta) = 1, \end{aligned} \tag{11}$$

and with appropriate initial conditions for f and θ.

3 PERTURBATION METHOD

The main goal of the study is to transform the PDE system into an ODE system that is easier to solve with semi-analytical techniques. To achieve that, we need to eliminate one of the two independent variables, x and η, of the initial PDE system. Assuming that the boundary layer is at the beginning of its formation, that x is small ($x << 1$) and can be the perturbation parameter of the problem under consideration, ε, we introduce the following expansion series for f and θ [3]:

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_n, \qquad \theta = \sum_{n=0}^{\infty} \varepsilon^n \theta_n, \tag{12}$$

and the expansions for the derivatives of f and θ, with respect to η and ε:

$$f^{(m)} = \sum_{n=0}^{\infty} \varepsilon^n f_n^{(m)}, \qquad \theta^{(m)} = \sum_{n=0}^{\infty} \varepsilon^n \theta_n^{(m)}, \tag{13}$$

$$\frac{\partial f^{(m)}}{\partial \varepsilon} = \sum_{n=0}^{\infty} n\varepsilon^{n-1} f_n^{(m)}, \qquad \frac{\partial \theta^{(m)}}{\partial \varepsilon} = \sum_{n=0}^{\infty} n\varepsilon^{n-1} \theta_n^{(m)}. \tag{14}$$

Introducing equations (12)-(14) to the system and boundary conditions (9)-(11) and separating orders of ε^0, ε^1 and ε^2, respectively, we obtain the ODEs [3]:

$$\varepsilon^0: \quad f_0''' + \frac{1}{2} f_0 f_0'' = 0, \tag{15}$$

$$\frac{1}{Pr}\theta_0'' + \frac{1}{2} f_0 \theta_0' + \frac{u_e^2}{C_p(T_w - T_e)} (f_0'')^2 = 0, \tag{16}$$

$$\varepsilon^1: \quad f_1''' + \frac{1}{2} f_0 f_1'' - f_0' f_1' + \frac{3}{2} f_1 f_0'' + \frac{1}{u_e}\frac{du_e}{d\epsilon}(1 - (f_0')^2) = 0, \tag{17}$$

$$\frac{1}{Pr}\theta_1'' + \frac{1}{2} f_0 \theta_1' - \theta_1 f_0' + \frac{16\alpha\sigma T_e^3}{\rho C_p u_e}(1 - \theta_0) - \frac{u_e}{C_p(T_w - T_e)}\frac{du_e}{d\epsilon} f_0' + \frac{3}{2} f_1 \theta_0' + 2\frac{u_e^2}{C_p(T_w - T_e)} f_0'' f_1'' = 0, \tag{18}$$

$$\varepsilon^2: \quad f_2''' + \frac{1}{2} f_0 f_2'' + \frac{5}{2} f_2 f_0'' - 2 f_0' f_2' + \frac{3}{2} f_1 f_1'' - (f_1')^2 - \frac{2}{u_e}\frac{du_e}{d\epsilon} f_0' f_1' = 0, \tag{19}$$

$$\begin{aligned} &\frac{1}{Pr}\theta_2'' + \frac{1}{2} f_0 \theta_2' - 2\theta_2 f_0' - \theta_1 f_1' + \frac{3}{2} f_1 \theta_1' - \frac{16\alpha\sigma T_e^3}{\rho C_p u_e}\theta_1 - \frac{u_e}{C_p(T_w - T_e)}\frac{du_e}{d\epsilon} f_1' + \\ &\frac{5}{2} f_2 \theta_0' + \frac{u_e^2}{C_p(T_w - T_e)} f_1'' + 2\frac{u_e^2}{C_p(T_w - T_e)} f_0'' f_2'' = 0, \end{aligned} \tag{20}$$

with boundary conditions:

$$\begin{aligned} \eta = 0: \quad & f_0 = f_1 = f_2 = 0, \quad f_0' = f_1' = f_2' = 0, \quad \theta_0 = \theta_1 = \theta_2 = 0, \\ \eta \to \delta: \quad & f_0' = 1,\ f_1' = f_2' = 0, \quad \theta_0 = 1,\ \theta_1 = \theta_2 = 0, \end{aligned} \tag{21}$$

The ODEs and boundary conditions (15)-(21) are functions only of η, $f(\eta)$ and $\theta(\eta)$. The initial PDE system was transformed into an ODE system. Equation (15) is a non-linear ODE, known as the Blasius equation and equations (16)-(20) are linear ODEs.

4 APPROXIMATE SEMI-ANALYTICAL SOLUTIONS

Two semi-analytical methods are used in this study, the homotopy analysis method (HAM) and the differential transformation method (DTM). These methods were utilized to examine the Blasius problem in several studies [7, 9, 8]. For the solution of the entire system of ODEs we used a hybrid method, where the zeroth order problems were solved using HAM and the rest (ε^1 and ε^2 problems), were solved with the DTM. For the solution of the Blasius problem:

$$\begin{cases} f_0''' + \dfrac{1}{2} f_0 f_0'' = 0, \\ \eta = 0 : \; f_0(0) = f_0'(0) = 0, \\ \eta \to \delta : \; f_0'(\eta) = 1, \end{cases} \tag{22}$$

we further obtain the approximate semi-analytical solution using HAM and DTM.

4.1 Homotopy Analysis Method

We introduce the linear operator $\mathcal{L} = \frac{\partial^3}{\partial \eta^3}$, and we construct a family of equations, called the zeroth-order deformation equation:

$$(1-p)[F'''(\eta,h,p) - f\,_0'''(\eta)] = ph\left[F'''(\eta,h,p) + \frac{1}{2}F(\eta,h,p)F''(\eta,h,p)\right] \tag{23}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is the convergence control parameter, $\eta \in [0,+\infty)$ and $f_0(\eta) = \frac{\sigma\eta^2}{2}$ the initial approximation, with corresponding boundary conditions at $\eta = 0$:

$$F[0,h,p] = F'[0,h,p] = 0, \quad F''[0,h,p] = \sigma \neq 0, \tag{24}$$

the prime denotes derivatives with respect to η [6, 7]. Differentiating m times equations (23) and (24) with respect to p, setting $p = 0$ and integrating with respect to η, the following mth-order approximation of $f_0(\eta)$ is:

$$f_{0,m}(\eta,h) = f_{0,0}(\eta) + \sum_{k=1}^{m} \frac{f_{0,k}(\eta,h)}{k!}, \quad \eta \in [0,+\infty), \quad h \neq 0. \tag{25}$$

Equating the terms of the above Taylor series with the terms of $\frac{\partial F}{\partial p}, \frac{\partial^2 F}{\partial p^2}, \cdots, \frac{\partial^m F}{\partial p^m}$, we get the general form:

$$f_{0,m}(\eta,h) = \sum_{k=0}^{m} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2} \Phi_{m,k}(h), \tag{26}$$

where $A_0 = 1$, $A_1 = 1$ and A_k is given by:

$$A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} \quad (k \geq 2), \tag{27}$$

and the approaching function $\Phi_{m,n}(h)$ is given by $\Phi_{m,n}(h) = 0$ for $n > m$, $\Phi_{m,n}(h) = 1$ for $n \leq 0$ and [7]:

$$\Phi_{m,n}(h) = (-h)^n \sum_{k=0}^{m-n} \binom{m}{m-n-k} \binom{n+k-1}{k} h^k, \qquad 1 \leq n \leq m. \tag{28}$$

4.2 Differential Transformation Method

The Blasius problem (22) is solved with DTM. According to the basic transformations the terms of Blasius equation become [9]:

$$\begin{cases} f'''(\eta) = (k+1)(k+2)(k+3)F(k+3), \\ \dfrac{1}{2} f(\eta) f''(\eta) = \dfrac{1}{2} \displaystyle\sum_{r=0}^{k} (k-r+1)(k-r+2)F(r)F(k-r+2). \end{cases} \tag{29}$$

Thus, the Blasius problem is transformed to:

$$F(k+3) = -\frac{\sum_{r=0}^{k}(k-r+1)(k-r+2)F(r)F(k-r+2)}{2(k+1)(k+3)(k+3)}, \tag{30}$$

and the boundary conditions are transformed into the following:

$$F(0) = 0,\ F'(0) = 0,\ F(2) = a. \tag{31}$$

Constant a, has been calculated as $a = \frac{\sigma}{2} = 0.16603$ [9], where, σ, is the constant that occurred form Howarth's numerical solution [2]. The dimensionless stream function, $f(\eta)$, will be given by:

$$f(\eta) = \sum_{k=0}^{m} \frac{F(k)\eta^k}{k!}. \tag{32}$$

To further improve the convergence of the solution (34), we use Padé approximants [10, 18].

4.3 Approximate Semi-analytical Solution of the Entire System of ODEs

For the zeroth order energy equation we use HAM and working similarly as with the Blasius equation, we get:

$$\theta_{0,m}(\eta,h) = b\eta + \sum_{k=1}^{m} b\left(\frac{Pr}{2}\right)^k \int\int f_0(\eta)\left[\frac{1}{(k-1)!}\left(\int f_0(\eta)d\eta\right)^k\right] d\eta d\eta \Psi_{m,k}(h), \tag{33}$$

where $\Psi_{m,n}(h)$ is the approaching function as described before [7]. Using DTM, the first and second order momentum and energy equations, (17)-(20), become:

$$\begin{aligned} F_1(k+3) = \Bigg(&-\frac{1}{2}f_0(\eta)(k+1)(k+2)F_1(k+2) + f_0'(\eta)(k+1)F_1(k+1) - \\ &-\frac{3}{2}f_0''(\eta)F_1(k) - \frac{1}{u_e}\frac{du_e}{d\varepsilon}[(f'_0)^2 - 1]\Bigg) \Big/ (k+1)(k+2)(k+3), \end{aligned} \tag{34}$$

$$\Theta_1(K+2) = \frac{Pr\left(-\frac{1}{2}f_0(\eta)(k+1)\Theta_1(k+1) + f_0'(\eta)\Theta_1(k) - C_1(\eta)\right)}{(k+1)(k+2)}, \tag{35}$$

$$\begin{aligned} F_2(k+3) = \Bigg(&-\frac{1}{2}f_0(\eta)(k+1)(k+2)F_2(k+2) + 2f_0'(\eta)(k+1)F_2(k+1) - \\ &-\frac{5}{2}f_0''(\eta)F_2(k) + C_2(\eta)\Bigg) \Big/ (k+1)(k+2)(k+3), \end{aligned} \tag{36}$$

$$\Theta_2(K+2) = \frac{Pr\left(-\frac{1}{2}f_0(\eta)(k+1)\Theta_2(k+1) + f_0'(\eta)\Theta_2(k) - C_3(\eta)\right)}{(k+1)(k+2)}, \tag{37}$$

respectively, and the non-homogeneous terms of the equations are:

$$C_1(\eta) = \frac{16\alpha\sigma T_e^3}{\rho C_p u_e}(1-\theta_0) - \frac{u_e}{C_p(T_w - T_e)}\frac{du_e}{d\epsilon}f_0' + \frac{3}{2}f_1\theta_0' + 2\frac{u_e^2}{C_p(T_w - T_e)}f_0''f_1'', \tag{38}$$

$$C_2(\eta) = -\frac{3}{2}f_1(\eta)f_1''(\eta) + (f_1'(\eta))^2 - \frac{2}{u_e}\frac{du_e(\varepsilon)}{d\varepsilon}f_0'(\eta)f_1'(\eta), \tag{39}$$

$$\begin{aligned} C_3(\eta) = &-\frac{16\alpha\sigma T_e^3}{\rho C_p u_e}\theta_1 - \theta_1 f_1' + \frac{3}{2}f_1\theta_1' - \frac{u_e}{C_p(T_w - T_e)}\frac{du_e}{d\epsilon}f_1' + \\ &+\frac{5}{2}f_2\theta_0' + \frac{u_e^2}{C_p(T_w - T_e)}f_1'' + 2\frac{u_e^2}{C_p(T_w - T_e)}f_0''f_2'' = 0. \end{aligned} \tag{40}$$

5 RESULTS & DISCUSSION

We compare the results given by Blasius semi-analytical solution, HAM, DTM and a numerical solution of the Blasius problem. As shown in Table 1, the numerical solution of the equation is very close to both HAM and DTM. The results are in accordance with Blasius' results, but the approximate semi-analytical methods achieve better convergence, especially after $\eta = 5.69$ that Blasius solution diverges. Homotopy analysis method (HAM) tends to slightly overestimate the produced results. Differential transformation method (DTM) with the selection of appropriate Padé approximants satisfies the boundary conditions e.g. at $\eta \to \infty$, $f'(\infty) = 1$. The numerical solution is the same with the produced semi-analytical results, with minor differences occurring in the seventh decimal digit. In Figure 1, the effects of adverse and favorable pressure gradient are presented. The adverse pressure gradient decreases the dimensionless velocity of the boundary layer, while favorable pressure gradient increases the dimensionless velocity, $f'(\eta)$. The approximate semi-analytical solution of the system overestimates the solution at some intervals, especially at $\eta \in [2, 4]$, compared with the numerical solution of the system. This occurs because HAM overestimates the solution at that interval. Despite that, the examination of the maximum error of the approximate semi-analytical with respect to the numerical solution of the dimensionless velocity, $f'(\eta)$, is 6.85% at $\eta = 0.5$, as shown in Table 2 for the entire system of ODEs.

$f'(\eta)$				
η	**Blasius**	**HAM**	**DTM**	**Numerical**
0.0	0.0	0.0	0.0	0.0
0.5	0.1658865	0.1658901	0.1658852	0.1658852
1.0	0.3297826	0.3298377	0.3297800	0.3297800
1.5	0.4867931	0.4870451	0.4867893	0.4867893
2.0	0.6297705	0.6304222	0.6297657	0.6297657
2.5	0.7512651	0.7523999	0.7512597	0.7512596
3.0	0.8460502	0.8474424	0.8460444	0.8460444
3.5	0.9130462	0.9142262	0.9130403	0.9130404
4.0	0.9555240	0.9561711	0.9555182	0.9555182
4.5	0.9795199	0.9798064	0.9795142	0.9795142
5.0	0.9915474	0.9919728	0.9915419	0.9915418
5.5	0.9997910	0.9977319	0.9968789	0.9968787
6.0	10^8	1.0000458	0.9989734	0.9989728

Table 1: Semi-analytical solution of the dimensionless velocity, $f'(\eta)$, utilizing the solution given by Blasius, HAM, DTM and a numerical method for the solution of Blasius' problem.

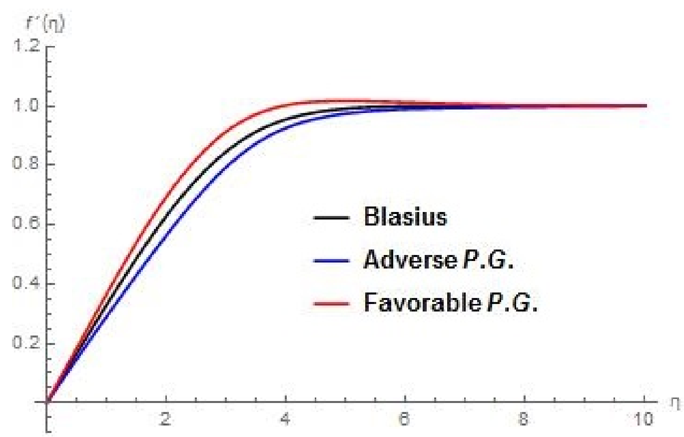

Figure 1: Effects of adverse (A.P.G) and favorable (F.P.G) pressure gradient on the dimensionless velocity $f'(\eta)$, for the entire momentum system for $\varepsilon = 0.06$.

The numerical solution of the energy equations is in accordance with the approximate semi-analytical solution of the system, as shown in Table 2. The maximum error of the approximate semi-analytical with respect to the numerical solution is 4.57%. The approximate semi-analytical solution is overestimated compared with the numerical solution, due to the use of the HAM, which is used to solve the zeroth order energy equation. In Figure 2, the combined effect of radiation and adverse pressure gradient is presented for the cases of cooling ($T_w < T_e$) and heating plate ($T_w > T_e$). In the case of a cooling plate there is a temperature increase in the boundary layer with the presence of adverse pressure gradient and radiation compared with the flow without. For the heating plate, the temperature decreases in the boundary layer when there is adverse pressure gradient and radiation.

	$f'(\eta)$			$\theta(\eta)$		
η	**Semi-Analytical**	**Numerical**	**Error %**	**Semi-analytical**	**Numerical**	**Error %**
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	0.1573318	0.1472349	6.857	0.1521036	0.1549216	1.819
1.0	0.3120108	0.2999781	4.011	0.3035827	0.3081596	1.485
1.5	0.4609632	0.4526817	1.829	0.4529625	0.4557061	0.602
2.0	0.5994947	0.5970037	0.417	0.5972717	0.5918522	0.915
2.5	0.7215306	0.7237476	0.306	0.7304469	0.7106039	2.792
3.0	0.8206348	0.8256130	0.602	0.8418044	0.8073880	4.262
3.5	0.8929787	0.8995961	0.735	0.9206946	0.8804197	4.574
4.0	0.9400546	0.9476863	0.805	0.9669564	0.9311014	3.850
4.5	0.9677283	0.9754799	0.794	0.9904395	0.9633034	2.816
5.0	0.9828953	0.9897055	0.688	1.0010844	0.9819859	1.944
5.5	0.9908735	0.9961404	0.528	1.0052037	0.9918682	1.344
6.0	0.9948568	0.9987106	0.385	1.0062189	0.9966304	0.962

Table 2: Comparison of the semi-analytical and numerical solution of the dimensionless velocity, $f'(\eta)$, and dimensionless temperature, $\theta(\eta)$, of the entire momentum and energy equations system and error of the semi-analytical with respect to the numerical solution.

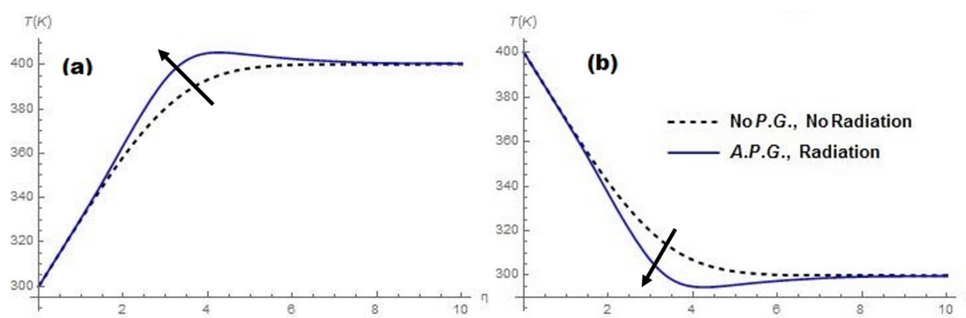

Figure 2: Radiation and adverse pressure gradient effects on the temperature of the boundary layer for a temperature difference between the flat plate and the fluid, $\Delta T = 100K$. The two cases are: (a) cooling plate and (b) heating plate.

6 CONCLUSIONS

In this study, we examined the incompressible, laminar boundary layer flow over a flat plate under the effects of pressure gradient and radiation. The governing equations of the problem were non-dimensionalized using the Falkner-Skan transformation. The PDE system that was obtained was transformed into an ODE system utilizing classical perturbation techniques. In the new system, the Blasius problem appears, for which solutions were obtained utilizing two approximate semi-analytical methods, HAM and DTM. The entire system was solved using HAM for the zeroth order equations and DTM for the first and second order equations. Adverse pressure gradient decreases the dimensionless velocity of the boundary-layer while favorable pressure gradient increases the dimensionless velocity. The combined effect of pressure gradient and radiation was also examined. Adverse pressure gradient and radiation increase the temperature in the case of a cooling plate, while adverse pressure gradient and radiation decrease the temperature of the boundary layer in the case of a heating plate.

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