8th GRACM International Congress on Computational Mechanics
Volos, 12 July – 15 July 2015

A MODIFIED DECOUPLED SCALED BOUNDARY-FINITE ELEMENT METHOD FOR MODELING 2D IN-PLANE-MOTION TRANSIENT ELASTODYNAMIC PROBLEMS IN SEMI-INFINITE MEDIA

Denise-Penelope N. Kontoni[1] and Mohammad Iman Khodakarami[2]

[1]Department of Civil Engineering, Technological Educational Institute of Western Greece,
1 M. Alexandrou Str., Koukouli, GR-26334 Patras, Greece
e-mail: kontoni@teiwest.gr ; web page: www.teiwest.gr

[2]Faculty of Civil Engineering, Semnan University,
Elm-o-SanatBlv., Semnan, 35131-19111, Iran
e-mail: khodakarami@semnan.ac.ir ; web page: www.semnan.ac.ir

Keywords: Two-dimensional elastodynamics, Decoupled "Scaled Boundary-Finite Element Method" (SBFEM), Semi-infinite media, In-plane waves.

Abstract. *In this paper a modified semi-analytical method based on scaled boundary approaches has been used for solving two-dimensional elastodynamic boundary value problems in semi-infinite domain; in this method, which is called Decoupled Scaled Boundary-Finite Element Method (DSBFEM), the boundary of the problem domain is discretized by specific higher-order sub-parametric elements; where, the Lagrange polynomials are implemented as mapping functions and the Gauss-Lobatto-Legendre (GLL) quadrature is used for calculating the coefficient matrices in this method. The special shape functions which are set up at GLL control points lead to diagonal coefficient matrices; so, the system of the governing partial differential equation in a given problem will be depend only upon the elastodynamic function of each degree of freedom (DOF). Here in, a classical in-plane surface exciting Lamb's problem is studied and the results are compared with the results from other solution methods. This study shows the DSBFEM is an efficient tool in order to analyze time-domain problems which are subjected to surface waves, and this method solves these kind of problems (which usually are studied in large-scale) with a few number of DOF and this will be considerable for researchers who work in this field.*

1 INTRODUCTION

To solve elastodynamic problems for analysis and design purposes, numerical approaches are usually employed. Different types of numerical approaches such as Finite Element Method (FEM), Boundary Element Method (BEM), Scaled Boundary Finite Element Method (SBFEM), and meshless methods are routinely used to solve elastodynamic problems. The use of FEM is advantageous as its procedures are well-established and versatile in nature. On the other hand, BEM requires basically reduced surface discretizations, and may be considered as an appealing alternative to FEM for elastodynamic problems. Since BEM does not require domain discretization, fewer unknowns are required to be stored. BEM needs a fundamental solution for the governing differential equation to obtain the boundary integral equation. Although coefficient matrices of BEM are much smaller than those of FEM, they are routinely non-positive definite, non-symmetric, and fully populated. Combining the advantages of FEM and BEM, SBFEM was successfully developed. Using surface finite elements, SBFEM discretizes only the boundary of the domain by transforming the governing partial differential equations to ordinary differential equations, which may be solved analytically. SBFEM, which requires no fundamental solution, have also been employed for the analysis of elastodynamic problems [1]. A modification of the SBFEM with Diagonal coefficient matrices has been proposed in [2] for solving potential problems and it is applied to solve elastostatic problems [3], also the proposed method is used to solve elastodynamic problems in [4], moreover it is also the method to solve the three-dimensional elastostatic problems [5] and an infinite half-space problems is used [6].

In the present paper, the application of a novel semi-analytical method for the analysis of wave propagation problem in a semi-infinite domain is investigated. Herein, we used a modification approach of the semi-analytical approach which is proposed in [4]; where, the Lagrange polynomials is used as mapping functions instead of Chebyshev polynomials and also Gauss-Lobatto-Legendre quadrature is employed instead of Clenshaw-Curtis integration technique in order to calculate the coefficient matrices. By the way, with implementing this technique, the governing equations for each node are independent of the other nodes and this will reduce the computational costs. Accuracy of the present method for solving surface wave propagation is demonstrated through a benchmark problem and its results are shown a good agreement between this approach and other methods.

2 SUMMARY OF DSBFEM

The governing equilibrium equations for elastodynamic problems may be solved based upon either a strong or a weak formulation of the problem. In the strong formulation, one may directly deal with the equilibrium equations of the problem and associated boundary conditions written in a differential form. In the weak formulation on the other hand, one may use an integral form of the equilibrium equations. The present new method is based on a weak formulation of the governing equations of elastodynamic problems. The equilibrium equations for a two-dimensional bounded medium Ω is described by

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \,, \tag{1}$$

in which, σ_{ij} indicates the stress tensor components. In addition, f_i denotes the external source of exciting forces generated per unit volume, respectively, $\ddot{u}_i = \partial^2 u_i / \partial t^2$ and ρ is the mass density. For a semi-infinite 2D problem in global Cartesian coordinates, $i = X, Y$ and $j = X, Y$ (see Figure 1).

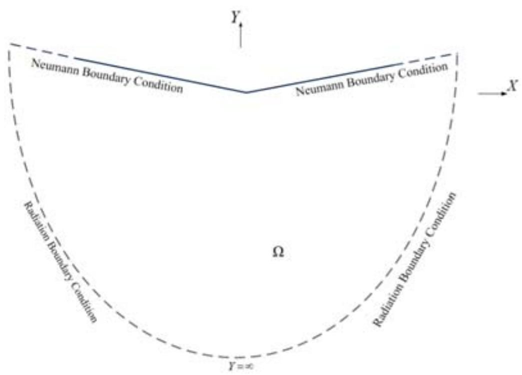


Figure 1. A semi-infinite 2D domain with related boundary conditions for elastodynamic problems.

Instead of using the governing equations and associated boundary conditions directly, strong form as in Eq. (1), one may use an integrated. This may be accomplished by weighting Eq. (1) with an arbitrary weighting function (w), and integrating over the problem domain Ω and applying the boundary conditions. The result may be given by

$$\int_{\Omega} w(\sigma_{ij,j} + f_i - \rho \ddot{u}_i)\, d\Omega = 0 \,, \tag{2}$$

In the present method, for a bounded medium, a local-coordinates-origin (LCO) is chosen from which all boundaries of the domain are visible (Figure 2). For bounded domains of this research, the LCO is selected inside the domain or on the boundary. Consequently, the total boundary of the domain includes two regions: the region that pass through the LCO (such as $\Gamma_1$and Γ_5 in Figure 2a), and the other remaining region. In the present method, only the region that does not pass through the LCO should be decomposed into n_e one-dimensional non-isoparametric elements Γ_e, $e = 1,2, \ldots, n_e$, so that $\Gamma = \cup_e^{n_e} \Gamma_e$.

In this method, a geometry transmission is introduced from global Cartesian coordinates $(\hat{x}, \hat{y})$ to local dimensionless coordinates (ξ, η) (see Figure 2). This transmission is obtained by Lagrange polynomials as mapping functions. Two local coordinates are defined as ξ is radial coordinate from the LCO to the boundaries and η is tangential coordinate on the boundaries. The radial coordinate ξ is equal to zero at the LCO and is equal to 1 on the free boundaries and is infinity at the boundaries with radiation condition. The tangential coordinate η varies between -1 and +1 on the boundaries. In addition, displacement and stress of each node are interpolated by special shape functions that are introduced in this paper. Mapping functions and the special shape functions are illustrated below.

Each element on the boundary is analogous to a line; so, we adopt an appropriate mapping between the line (reference/ master element) and each element Γ_e. In the present method, sub-parametric elements are proposed. Each element Γ_e is defined in terms of a set of mapping functions $\Phi_i(\eta), i = 1,2,\ldots, n_\eta + 1$. The geometry of elements in local coordinates is then given as

$$\{x(\eta)\} = [\Phi(\eta)]\{x\}, \tag{3}$$

where $\{x(\eta)\}$ denote the vector of global coordinates of boundary points; and any point in the domain with $\{\hat{x}\}$ coordinates relates to the corresponding point on the elements of boundary as

$$\{\hat{x}(\xi,\eta)\} = \xi\{x(\eta)\} \tag{4}$$

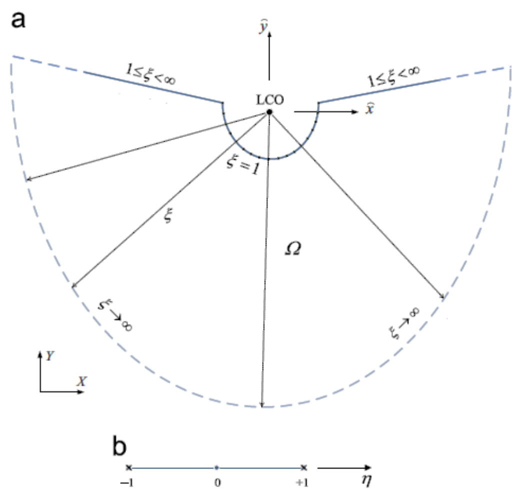


Figure 2. Modeling of 2D semi-infinite domain, and the LCO: (a) in global coordinates system, and (b) in scaled boundary local system.

The proposed mapping functions for an$(n_\eta + 1)$-node element is introduced as follows[7]

$$\phi_i(\eta) = \prod_{k=1,k\neq i}^{n_\eta+1} \frac{\eta-\eta_k}{\eta_i-\eta_k} \tag{5}$$

Considering Eq. (5), the Lagrange polynomials have the properties of the Kronecker Delta at any G-L-L node of the elements on the boundaries. To this end, let's consider a differential element of area $d\hat{x}d\hat{y}$ which is related to a differential element of area $d\xi d\eta$ by

$$d\Omega = d\hat{x}d\hat{y} = \left|\hat{J}(\xi,\eta)\right| d\xi d\eta = \xi|J(\eta)|d\xi d\eta \tag{6}$$

where, $\hat{J}(\xi,\eta)$ indicates the 2×2 Jacobian matrix of the transformation, and is evaluated as

$$\hat{J}(\xi,\eta) = \left\|\frac{\partial\vec{\hat{x}}}{\partial\xi}\times\frac{\partial\vec{\hat{x}}}{\partial\eta}\right\|; \quad \vec{\hat{x}} = \begin{Bmatrix}\hat{x}\\ \hat{y}\end{Bmatrix}. \tag{7}$$

On the boundary, the Jacobian matrix may be written in the following form

$$J(\eta) = \begin{bmatrix} x(\eta) & y(\eta) \\ x_{,\eta}(\eta) & y_{,\eta}(\eta)\end{bmatrix} \tag{8}$$

The spatial derivatives of a virtual vector $\{S\} = [S_{\hat{x}} S_{\hat{y}}]^T$ in global coordinates system are defined in terms of the virtual vector by the well-known relation that defines the operator [L]

$$\begin{Bmatrix} S_{,\hat{x}} \\ S_{,\hat{y}} \\ S_{,\hat{x}} + S_{,\hat{y}} \end{Bmatrix} = \begin{bmatrix} \partial/\partial \hat{x} & 0 & \partial/\partial \hat{y} \\ 0 & \partial/\partial \hat{y} & \partial/\partial \hat{x} \end{bmatrix}^T \begin{Bmatrix} S_{\hat{x}} \\ S_{\hat{y}} \end{Bmatrix} \tag{9}$$

The spatial derivatives in the two coordinate systems are related as

$$\begin{bmatrix} \partial/\partial \hat{x} & 0 & \partial/\partial \hat{y} \\ 0 & \partial/\partial \hat{y} & \partial/\partial \hat{x} \end{bmatrix}^T = [b^1(\eta)]\frac{\partial}{\partial \xi} + [b^2(\eta)]\frac{1}{\xi}\frac{\partial}{\partial \eta}, \tag{10}$$

where,

$$[b^1(\eta)] = \frac{1}{|J(\eta)|}\begin{bmatrix} y(\eta)_{,\eta} & 0 \\ 0 & -x(\eta)_{,\eta} \\ -x(\eta)_{,\eta} & y(\eta)_{,\eta} \end{bmatrix} \tag{11}$$

$$[b^2(\eta)] = \frac{1}{|J(\eta)|}\begin{bmatrix} -y(\eta) & 0 \\ 0 & x(\eta) \\ x(\eta) & -y(\eta) \end{bmatrix} \tag{12}$$

In this method, special polynomials N (η) are used as shape functions, diagonal coefficient matrices will be derived by using these shape functions for solving problems. To this end, the displacement function and its derivatives, across the element, are interpolated using polynomials that own two specific characteristics; the shape functions have Kronecker Delta property, and their first derivatives are equal to zero at any given control point. For an element with (n_η+1) nodes, these shape functions are expressed as a polynomial of degree ($2n_\eta$+1) as the following form [3]

$$N_i(\eta) = \sum_{m=0}^{2n_\eta+1} a_m \eta^m \tag{13}$$

The displacement field $\{u(\xi,\eta,t)\}$ at any point (ξ,η) and time t is obtained by interpolation of the displacement function using shape functions. For a two dimensional problem, the displacement field may be written in the following form

$$\{u(\xi,\eta,t)\} = \lfloor u_x(\xi,t) \quad u_y(\xi,t) \rfloor^T \tag{14}$$

Using Eqs. (10) and (14), the strain vector at a given point (ξ,η) at time t will be obtained by

$$\{\varepsilon(\xi,\eta,t)\} = [B^1(\eta)]\{u(\xi,t)\}_{,\xi} + \frac{1}{\xi}[B^2(\eta)]\{u(\xi,t)\}\ , \tag{15}$$

where, $[B^1(\eta)] = [b^1(\eta)][N(\eta)]$ and $[B^2(\eta)] = [b^2(\eta)][N(\eta)]_{,\eta}$. As the first derivatives of the shape functions at nodes are zero, the second term of Eq. (15) at the G-L-L nodes will be vanished. The relation between strain and stress may be expressed using Hook's Law and Eq. (15) using the elasticity matrix[D] as given by

$$\{\sigma(\xi,\eta,t)\} = [D]\big([b^1(\eta)][N(\eta)]\{u(\xi,t)\}_{,\xi} + [b^2(\eta)][N(\eta)]_{,\eta}\{u(\xi,t)\}\big). \tag{16}$$

As mentioned before, the weak form of governing equations of elastodynamic problems in global coordinates is expressed as Eq. (2). Implementing Eqs. (10), (16) into Eq. (2), leads to

$$\xi[D^0]\{u(\xi,t)\}_{,\xi\xi} + [D^1]\{u(\xi,t)\}_{,\xi} + \xi\{F^b(\xi,t)\} = \xi[M]\{u(\xi,t)\}_{,tt} \tag{17}$$

where,

$$[D^0] = \int_{-1}^{+1} [B^1(\eta)]^T [D][B^1(\eta)]|J(\eta)|d\eta\,, \tag{18}$$

$$[D^1] = \int_{-1}^{+1} [B^1(\eta)]_{,\eta}^T [D][B^2(\eta)]|J(\eta)|d\eta \tag{19}$$

$$[M] = \int_{-1}^{+1} [N(\eta)]^T \rho [N(\eta)]|J(\eta)|\, d\eta \tag{20}$$

$$\{F^b(\xi,t)\} = \int_{-1}^{+1} [N(\eta)]^T \{f^b(\xi,\eta,t)\}|J(\eta)|\, d\eta \tag{21}$$

Eq. (17) is the system of partial differential equations of radial coordinate ξ and time t which represents the governing equation of the present method for elastodynamic problems.

In this study, to calculate the vectors and matrices in Eq. (17), the Gauss-Lobatto-Legendre numerical integration method is applied. The numerical integration method, calculates the values of the coefficients matrix in the GLL, according to the node element that corresponds to the points and also features a shape functions used, resulting diagonal matrix of coefficients used in the equation. Weight coefficients used in the method of integration is calculated using the following equation [7]

$$w_i = \frac{2}{n(n+1)(n_\eta(\eta_i))^2} \tag{22}$$

Consequently, the components of coefficient matrices may be expressed as

$$D_{ij}^0 = \delta_{ij} w_i [B^1(\eta_i)]^T [D][B^1(\eta_i)]|J(\eta_i)| \tag{23}$$

$$D_{ij}^1 = \delta_{ij} w_i [B^1(\eta_i)]_{,\eta}^T [D][B^2(\eta_i)]|J(\eta_i)| \tag{24}$$

$$M_{ij} = \delta_{ij} w_i [N(\eta_i)]^T \rho [N(\eta_i)]|J(\eta_i)| \tag{25}$$

where, δ_{ij} denotes the Kronecker Delta which results in diagonal coefficient matrices. Consequently, the set of differential Eq. (17) may be expressed as a single differential equation regarding to a specified point i as the following expression

$$\xi D_{ii}^0 u(\xi,t)_{i,\xi\xi} + D_{ii}^1 u(\xi,t)_{i,\xi} + \xi F_i^b(\xi,t) = \xi M_{ii} u(\xi,t)_{i,tt} \tag{26}$$

It is worthwhile remarking that Eq. (26) offers a set of ordinary differential equations for an elastodynamic problem with 2*n* DOFs. Each differential equation in Eq. (26) depends only on the elastodynamic function of the *i*th DOF. This means that the coupled system of differential equations has been transformed into decoupled differential equations using a special set of weak formulation procedure, mapping functions, quadrature, and shape functions. In other words, to evaluate the displacement function and its derivatives at a given point, the governing equation that is corresponding to the point should be solved, only as may be illustrated later, the decoupled differential equations system proposed in this paper may also provide higher rates of convergence by employing a few numbers of DOFs compared to other numerical methods.

There are many procedures available in the literature for solving Eq. (26), among which Fourier transform technique along with the separation of variables could be employed as adopted in this research. For the *i*th DOF, the solution of Eq. (26) is assumed as the following form

$$u_i(\xi,t) = \Xi_i(\xi) T_i(t) \tag{27}$$

where, $\Xi_i(\xi)$ and $T_i(t)$ are functions of radial direction (ξ) and time (t), respectively. At first, $\Xi_i(\xi)$ may be determined by solving the homogenous form of partial differential Eq. (26) as

$$\xi D_{ii}^0 \Xi_i''(\xi) T_i(t) + D_{ii}^1 \Xi_i'(\xi) T_i(t) - \xi M_{ii} \Xi_i(\xi) \ddot{T}_i(t) = 0 \tag{28}$$

and,

$$M_{ii} \ddot{T}_i(t) = \omega_i^2 T_i(t) \tag{29}$$

in which, ω_i is the eigenvalue of the ith DOF. The Fourier transform of Eq. (26) using the convolution technique is written as

$$\mathcal{F}\{\xi * D_{ii}^{0} u(\xi,t)_{i,\xi\xi}\} + \mathcal{F}\{D_{ii}^{1} u(\xi,t)_{i,\xi}\} + \mathcal{F}\{\xi * F_i^{b}(\xi,t)\} = \mathcal{F}\{\xi * M_{ii} u(\xi,t)_{i,tt}\} \tag{30}$$

where, $\boldsymbol{\mathcal{F}}$ denotes the Fourier transform sign, and $*$ represents the convolution sign. Now, Eq. (30) may be rewritten as

$$-D_{ii}^{0}\omega_i^2 v(\omega_i,t)\mathcal{F}\{\xi\} + \frac{I}{\sqrt{2\pi}} D_{ii}^{1}\omega_i v(\omega_i,t) + \mathcal{F}\{\mathcal{F}_i^{b}(\xi,t)\}\mathcal{F}\{\xi\} = M_{ii} v(\omega_i,t)_{,tt}\mathcal{F}\{\xi\} \tag{31}$$

in which, $v(\omega_i,t)$ indicates the Fourier transform of $u(\xi,t)_i$ with respect to ξ, and I represents $\sqrt{-1}$. Moreover

$$\mathcal{F}\{\xi\} = \frac{(1 - e^{I\omega_i} + I\omega_i)e^{-I\omega_i}}{\omega_i^2\sqrt{2\pi}} \tag{32}$$

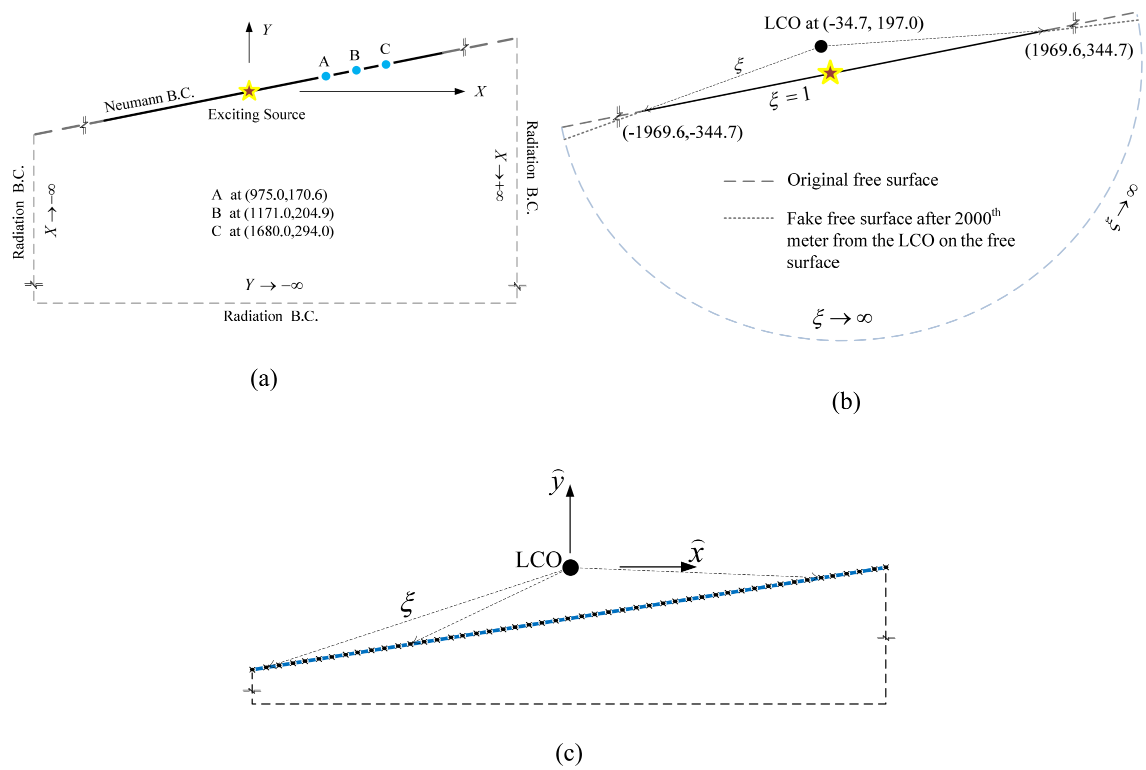
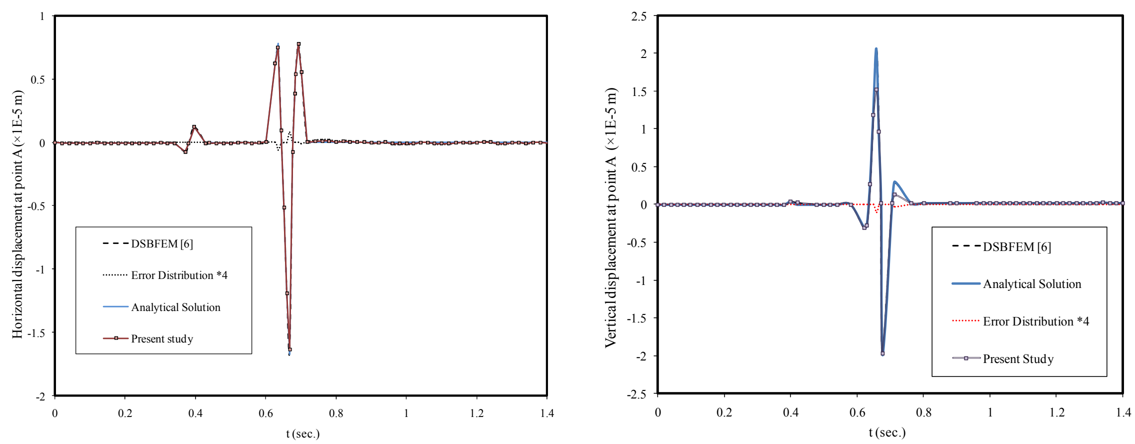


Figure 3. Modeling of the Lamb's problem; (a) the geometry of the domain, (b) the model of the free surface and far field boundaries in the local coordinate system, and the location of the LCO, (c) discretizing the free surface boundary in the local coordinate system using 50 three-node elements.

As there are many eigenvalues obtained by solving the ordinary differential Eq. (31) using related boundary conditions, the general solution of the governing partial differential equation will be represented as

$$u_i(\xi,t) = \sum_{n_\omega=1}^{N_\omega} \Xi_{i,n_\omega}(\xi) T_{i,n_\omega}(t) \tag{33}$$

where, N_ω represents the number of eigenvalues considered in each analysis. The proposed method obviously is a semi-analytical solution, which represents approximate numerical solutions on the boundaries, while offering exact analytical solutions within the undertaken domain of the problem.



(a) (b)

Figure 4. Time histories of displacement components at point A; (a) vertical component, and (b) horizontal component.

3 NUMERICAL EXAMPLE

In order to assess the accuracy of the DSBFEM for modeling of wave propagation problems in semi-infinite domains, a classic in-plane wave propagation problem with smooth free surface subjected to dynamic displacement excitations at the surface is considered, which, this problem is known as Lamb's problem.

The Lamb's problem [7] is a classical wave propagation problem which may show the accuracy of the solution methods used for solving seismological problems. This problem may be considered as a very good criterion for evaluating the abilities of a solution method for satisfying the free surface traction as well as radiation boundary condition in simulating strong surface waves. The geometry of the problem domain is shown in Figure 3a. The material properties of this example are: the Young's modulus $E = 1.88\times10^{10}$ kg/m^2, the Poisson's ratio $\nu = 0.25$, the mass density $\rho = 2200$ kg/m^3. With these material properties, the primary and secondary wave propagation velocities are 3200 and 1847.5 m/s, respectively.

As shown in Figure 3a, the free surface of the domain has 17.5% slope with respect to horizontal plane. The domain of this problem is subjected to a normal displacement surface source at ($X = 0$, $Y = 0$) whose time variation is expressed by a Ricker wavelet function with predominant frequency of 14.5Hz. The location of the LCO is shown in Figure 3b. In this situation, after a distance around 2000*m* at each side of the LCO, the free surface is approximately along the radial coordinate ξ (see Figure 3b). Consequently, it is not necessary to discretize the boundaries after these points (Figure 3c). As the governing equations of the DSBFEM are solved an analytically, the radiation condition at infinity will be automatically satisfied. In other words, no artificial far field BC is required to model the radiation condition at infinity.

The components of displacement at points A, B, and C are plotted in Figures 4-6. Moreover, the results obtained by analytical method by Lamb 1904 are depicted in these figures. As may be observed from the relative error between these two results, there are good agreement between the results of two methods. It is emphasizing that this problem has been solved using only 50 three-node elements (202 DOFs). As it is expected in plane strain problems, the surface waves (Rayleigh waves in this example) have constant amplitude along the free surface. This property of surface waves is clearly observable in these figures.

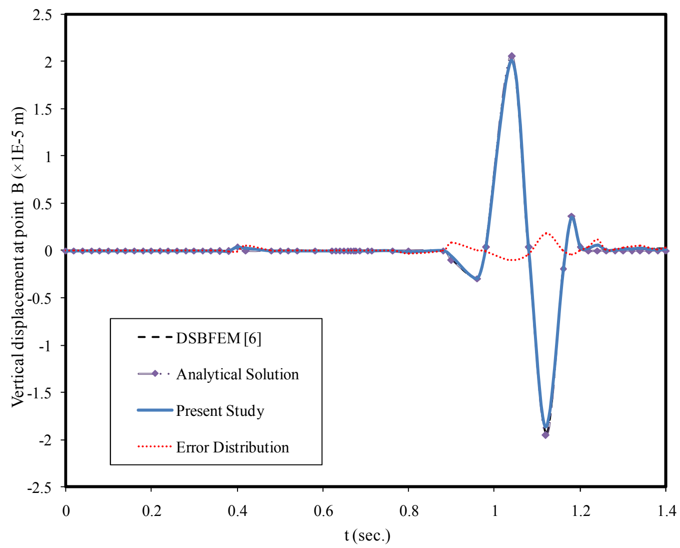


Figure 5. Time histories of vertical displacement components at point B.

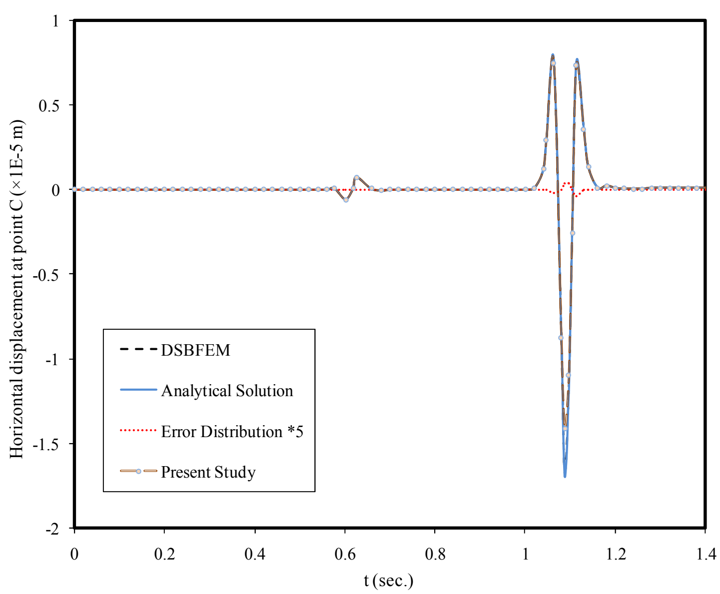


Figure 6. Time histories of horizontal displacement components at point C.

4 CONCLUSIONS

In this paper, a brief review on the formulation of the modified Decoupled Scaled Boundary Finite Element Method (DSBFEM), a novel semi-analytical method for modeling 2D wave propagation problems, has been summarized. Using the proposed sub-parametric elements, only the problem boundary at the free surface of a half-space domain should be discretized. The DSBFEM leads to a set of decoupled partial differential equations for solving the whole domain. In other words, the governing partial differential equation for each DOF has become independent from other DOFs of the domain. Time- and frequency-domain analysis of the classical benchmark problem as known as Lamb's problem has been successfully carried out using the DSBFEM. In this example, an in-plane wave propagation problem has been selected to illustrate the applicability and robustness of the DSBFEM for solving problems which are subjected to surface waves. It is worthwhile remarking that this problem is modeled with very small number of DOFs, while preserving high accuracies comparing with other analytical and numerical solutions.

REFERENCES

[1] Wolf, J.P. (2003), *The Scaled Boundary Finite Element Method*. John Wiley & Sons Ltd, Chichester, England.

[2] Khaji, N., Khodakarami, M.I. (2011), "A new semi-analytical method with diagonal coefficient matrices for potential problems", *Engineering Analysis with Boundary Elements,* Vol. 35(6), pp. 845-854.

[3] Khodakarami, M.I., Khaji, N. (2011), "Analysis of elastostatic problems using a semi-analytical method with diagonal coefficient matrices", *Engineering Analysis with Boundary Elements*, Vol. 35(12), pp. 1288-1296.

[4] Khodakarami, M.I., Khaji, N., Ahmadi, M.T. (2012), "Modeling transient elastodynamic problems using a novel semi-analytical method yielding decoupled partial differential equations", *Computer Methods in Applied Mechanics and Engineerin*g, Vol. 213-216, pp. 183-195.

[5] Khaji, N., Khodakarami, M.I. (2012), "A semi-analytical method with a system of decoupled ordinary differential equations for three-dimensional elastostatic problems", *International Journal of Solids and Structures*, Vol. 49, pp. 2528-2546.

[6] Khodakarami, M.I., Khaji, N. (2014), "Wave propagation in semi-infinite media with topographical irregularities using Decoupled Equations Method", *Soil Dynamics and Earthquake Engineering*, Vol. 65, pp.102-112.

[7] Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang, T.A. (1988), *Spectral Methods in Fluid Dynamics,* Springer-Verlag, Berlin.

[8] Lamb, H. (1904), "On the propagation of tremors over the surface of an elastic solid", *Philos. Trans. R. Soc. London, Ser. A*, Vol. 203, pp. 1–42.