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AN A POSTERIORI ERROR ANALYSIS FOR INTERIOR PENALTY DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD OF A TWO-DIMENSIONAL PROBLEM OF STRAIN GRADIENT ELASTICITY

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Abstract: *This work revolves around an a posteriori error analysis for (symmetric) Interior Penalty Discontinuous Galerkin Finite Element Method (IPDGFEM) of a two-dimensional problem of Strain Gradient Elasticity (SGE). Initially, we introduce the boundary value problem consisting of a system of partial differential equations and of essential boundary conditions. A reliable a posteriori error estimate of residual type is subsequently established in the energy seminorm for the (symmetric) IPDG method for that boundary value problem under minimal regularity assumptions on the analytical solution. Stability bounds of lifting operators are then deduced for a boundary value problem of SGE and by employing these stability bounds, we prove the coercivity and the continuity of bilinear form. As a result, a technical lemma of a recovery operator is presented for that boundary value problem using macro-elements. The reliable a posteriori error estimate, referred above, is based on a suitable recovery operator that maps discontinuous finite element spaces to C^1-conforming finite element spaces, consisting of triangles or quadrilateral macro-elements.*

1 INTRODUCTION

In this work, the main goal is to conduct an error analysis for IPDGFEM. Specifically, overall our research endeavor focuses on the introduction of a suitable recovery operator, on the proof of an appropriate Lemma for this operator and on the proof of h-version reliable a posteriori error estimate in the energy seminorm, $|||\cdot|||_{sg}$, for the IPDG method. Our analysis will be based on the ideas of Georgoulis and Houston in [11, 12] and our deducing proofs will contain many features presented therein. For that reason, additional care and considerations have to be employed in order to adapt, or in other words extend, their arguments to the problem of that work. The definitions of lifting operators, contained in Georgoulis and Houston [11], are extended for a higher dimensional boundary value problem in SGE (a system of partial differential equations supplemented with essential boundary conditions found in section 3). What is more, a reliable a posteriori error estimate of residual type is established in the energy seminorm for the (symmetric) IPDG method for problem of SGE, with essential boundary conditions under minimal regularity assumptions on the analytical solution. In section 5, stability bounds of lifting operators are deduced for a boundary value problem of SGE and by employing these stability bounds, we prove the coercivity and the continuity of bilinear form. As a result, in section 6, a technical lemma of a recovery operator is presented for that boundary value problem using macro-elements, by generalizing the results from [12] in vector spaces. The reliability estimate is based on a suitable recovery operator, that maps discontinuous finite element spaces to H_0^2-conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral macro-elements defined in [7] (see also [14, 3, 13, 12] for similar constructions). Also in section 6, using the recovery operator, in conjuction with the incosistent formulation for the IPDG method (which ensures that the weak formulation of the problem is defined under minimal regularity assumptions on the analytical solution), a reliable a posteriori error estimate of residual type can derive for the IPDG method in the corresponding energy seminorm.

2 PRELIMINARIES

Let us assume that the subdivision $\mathcal{T}$ is shape-regular (see p. 124 in [6]). We also assume that $\mathcal{E}$ is decomposed into two subsets, namely $\mathcal{E}_{\text{int}}$ and $\mathcal{E}_c$, which contain the set of all elements of $\mathcal{E}$ that are not subsets of Γ_c and the set of all elements of $\mathcal{E}$ that are subsets of Γ_c respectively, i.e.,

$$\mathcal{E}_{\text{int}} = \{e \in \mathcal{E} : e \subset \Omega\}, \quad \mathcal{E}_c = \{e \in \mathcal{E} : e \subset \Gamma_c\}.$$

We define the set Γ_{int} together with Γ_c as

$$\Gamma_{\text{int}} := \bigcup_{e \in \mathcal{E}_{\text{int}}} e, \quad \Gamma_c := \bigcup_{e \in \mathcal{E}_c} e,$$

all with the obvious meanings, respectively.
Let $\Gamma_0 = \Gamma_{\text{int}} \cup \Gamma_c$. We define for $\mathbf{u}, \mathbf{w} \in L^2(\Gamma_0)^2$, the inner product

$$\int_{\Gamma_0} \mathbf{u}\mathbf{w}dr = \int_{\Gamma_{\text{int}}} \mathbf{u}\mathbf{w}dr + \int_{\Gamma_c} \mathbf{u}\mathbf{w}dr \tag{1}$$

For each face $e \in \mathcal{E}_{\text{int}}$, let i and j be such indices that $i > j$ and the elements $K := K_i$ and $K' := K_j$ share the face e. Let us define the jump across e and the mean value on e of $\mathbf{u} \in H^1(\Omega, \mathcal{T})^2$ by

$$[\![\mathbf{u}]\!]_e := \mathbf{u}|_{\partial K \cap e} - \mathbf{u}|_{\partial K' \cap e} \quad \text{and} \quad \langle \mathbf{u} \rangle_e := \frac{1}{2}\left(\mathbf{u}|_{\partial K \cap e} + \mathbf{u}|_{\partial K' \cap e}\right),$$

respectively.
For the sake of convenience, we extend the definitions of the jump and of the mean value to faces $e \in \mathcal{E}_c$ by letting:

$$[\![\mathbf{u}]\!]_e = \mathbf{u}|_e \quad \text{and} \quad \langle \mathbf{u} \rangle_e = \mathbf{u}|_e.$$

With each face $e \in \mathcal{E}_{\text{int}}$ we associate the unit normal vector $n = n_{K_i}$ to e, pointing from element K_i to K_j when $i > j$, and with each $e \in \mathcal{E}_c$ we associate the external unit normal vector $n = n_K$, where $e \subset \partial K$.

3 MODEL PROBLEM

Toupin and Mindlin included higher-order stresses and strains in the theory of linear elasticity, which serves today as the foundation of more advanced strain gradient elasticity and plasticity formulation [17, 15], respectively. Let us introduce a two-dimensional model problem following their concepts.
Let Ω be a bounded, open, polygonal domain in $\Re^2$ and Γ_c its boundary. Let also Γ_∂ signify the union of one-dimensional open edges of Ω. The mechanical framework that we consider is strain gradient elasticity (or in other words dipolar gradient elasticity). The material constituting the structure is assumed to be isotropic, centrosymmetric and simplified.
We consider the equation:

$$(\lambda + \mu)\left\{\left(gD^2\right)^2 \mathbf{u} - D^2\mathbf{u}\right\} + \mu\left(g^2\Delta^2\mathbf{u} - \Delta\mathbf{u}\right) = \mathbf{f} - \Phi\nabla \quad \text{in } \Omega$$

or

$$(\lambda + \mu)D^2\left(g^2D^2\mathbf{u} - \mathbf{u}\right) + \mu\Delta\left(g^2\Delta\mathbf{u} - \mathbf{u}\right) = \mathbf{f} - \Phi\nabla \quad \text{in } \Omega, \tag{2}$$

where $\mathbf{f} - \Phi\nabla \in L^2(\Omega)^2$. In the above, D^2 is the symmetric Hessian matrix, Δ^2 is the biharmonic operator, Δ is the Laplace operator and $\mathbf{u}$ denotes the displacement field. In addition, we supplement the equation with the following boundary conditions

$$\begin{aligned} \mathbf{u} = \mathbf{0} \quad &\text{on } \Gamma_c, \\ \nabla\mathbf{u} \cdot n = \mathbf{0} \quad &\text{on } \Gamma_c, \end{aligned} \tag{3}$$

where n is the unit normal to the boundary, exterior to Ω. We mention that the boundary conditions are called homogeneous essential. Also notice that by construction Γ_c differs from Γ_∂ on a set of one-dimensional measure zero which contains the vertices of the (polygonal) boundary of Ω.

4 FINITE ELEMENT SPACES

For a non-negative integer p, we denote by $\mathcal{Q}_p(\hat{K})$ the set of all tensor product polynomials on $\hat{K}$ of degree at most p in each coordinate direction if $\hat{K}$ is the reference quadrilateral. We collect the h_K and p_K into the element-wise constant functions

$$\mathbf{h}, \mathbf{p} : \Omega \to \Re, \text{ with } \mathbf{h}|_K = h_K \text{ and } \mathbf{p}|_K = p_K, \ K \in \mathcal{T},$$

respectively. We consider the finite element space

$$\mathcal{S}_1 \equiv \mathcal{S}^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})^2 := \left\{ \mathbf{u} \in L^2(\Omega)^2 : \mathbf{u}|_K \circ F_K \in \mathcal{Q}_{p_K}(\hat{K})^2, \ K \in \mathcal{T} \right\}. \tag{4}$$

We shall assume throughout that the mesh size function $\mathbf{h}$ and polynomial degree function $\mathbf{p}$, with $p_K \geqslant 2$ for each $K \in \mathcal{T}$, have bounded local variation.

5 DGFEM with LIFTING OPERATORS

We would like to present the interior penalty discontinuous Galerkin method by using appropriate lifting operators for the problem (2) – (3). We shall employ the finite element space $\mathcal{S}_1$ constructed in the above section.
Let us first introduce the following functional space

$$H_0^2(\Omega)^2 = \left\{ \mathbf{u} | \ \mathbf{u} \in H^2(\Omega)^2 : \mathbf{u} = \mathbf{0}, \ \nabla \mathbf{u} \cdot n = \mathbf{0} \text{ on } \Gamma_c \right\}, \tag{5}$$

which is equipped with the norm induced by the Sobolev space $H^2(\Omega)^2$.
Next, we introduce the lifting operators $\mathcal{L}_i : \mathcal{S}^2 := \mathcal{S}_1 + H_0^2(\Omega)^2 \to \mathcal{S}_1, i = 1, 2, 3, 4$ by

$$\int_\Omega \mathcal{L}_1(\mathbf{u})\mathbf{w} dv = \int_{\Gamma_0} \left([\![\mathbf{u}]\!] \langle \nabla \mathbf{w} \rangle - \langle \mathbf{w} \rangle [\![\nabla \mathbf{u}]\!] \right) dr \quad \forall \mathbf{w} \in \mathcal{S}_1, \tag{6}$$

$$\int_\Omega \mathcal{L}_2(\mathbf{u})\mathbf{w} dv = \int_{\Gamma_0} [\![\mathbf{u}]\!] \langle \mathbf{w} \rangle dr \quad \forall \mathbf{w} \in \mathcal{S}_1, \tag{7}$$

$$\int_\Omega \mathcal{L}_3(\mathbf{u})\mathbf{w} dv = \int_{\Gamma_0} \left([\![\mathbf{u}]\!] \langle \nabla \mathbf{w} \rangle - \langle \mathbf{w} \rangle [\![\nabla \mathbf{u}]\!] \right) dr \quad \forall \mathbf{w} \in \mathcal{S}_1, \tag{8}$$

and

$$\int_\Omega \mathcal{L}_4(\mathbf{u})\mathbf{w} dv = \int_{\Gamma_0} [\![\mathbf{u}]\!] \langle \mathbf{w} \rangle dr \quad \forall \mathbf{w} \in \mathcal{S}_1. \tag{9}$$

The bilinear form $B_{sg} : \mathcal{S}^2 \times \mathcal{S}^2 \to \Re$ is defined as

$$\begin{aligned} B_{sg}(\mathbf{u}, \mathbf{w}) \quad := \quad & \int_\Omega (\lambda + \mu) g^2 D_h^2 \mathbf{u} D_h^2 \mathbf{w} dv + \int_\Omega (\lambda + \mu) \nabla_h \mathbf{u} \cdot \nabla_h \mathbf{w} dv \\ & + \int_\Omega \mu g^2 \Delta_h \mathbf{u} \Delta_h \mathbf{w} dv + \int_\Omega \mu \nabla_h \mathbf{u} : \nabla_h \mathbf{w} dv \\ & + \int_\Omega \left\{ (\lambda + \mu) g^2 D_h^2 \mathbf{u} \mathcal{L}_1(\mathbf{w}) + \mathcal{L}_1(\mathbf{u}) (\lambda + \mu) g^2 D_h^2 \mathbf{w} \right\} dv \\ & - \int_\Omega \left\{ (\lambda + \mu) \nabla_h \mathbf{u} \mathcal{L}_2(\mathbf{w}) + \mathcal{L}_2(\mathbf{u}) (\lambda + \mu) \nabla_h \mathbf{w} \right\} dv \\ & + \int_\Omega \left\{ \mu g^2 \Delta_h \mathbf{u} \mathcal{L}_3(\mathbf{w}) + \mathcal{L}_3(\mathbf{u}) \mu g^2 \Delta_h \mathbf{w} \right\} dv \\ & - \int_\Omega \left\{ \mu \nabla_h \mathbf{u} \mathcal{L}_4(\mathbf{w}) + \mathcal{L}_4(\mathbf{u}) \mu \nabla_h \mathbf{w} \right\} dv \\ & + \int_{\Gamma_0} \gamma [\![\mathbf{u}]\!] [\![\mathbf{w}]\!] dr + \int_{\Gamma_0} \zeta [\![\nabla \mathbf{u} \cdot n]\!] [\![\nabla \mathbf{w} \cdot n]\!] dr + \int_{\Gamma_0} \xi [\![\mathbf{u}]\!] [\![\mathbf{w}]\!] dr \\ & + \int_{\Gamma_0} \alpha [\![\mathbf{u}]\!] [\![\mathbf{w}]\!] dr + \int_{\Gamma_0} \beta [\![\nabla \mathbf{u} \cdot n]\!] [\![\nabla \mathbf{w} \cdot n]\!] dr + \int_{\Gamma_0} \delta [\![\mathbf{u}]\!] [\![\mathbf{w}]\!] dr, \end{aligned} \tag{10}$$

for any $\mathbf{u}, \mathbf{w} \in \mathcal{S}^2$, where D_h^2 defines the broken Hessian matrix, ∇_h defines the broken divergence (second integral) as well as the broken gradient (fourth integral) and Δ_h defines the broken Laplacian with respect to the

subdivision $\mathcal{T}$, respectively.
The linear form $L_{sg} : \mathcal{S}^2 \to \Re$ is given by

$$L_{sg}(\mathbf{w}) := \int_{\Omega} (\mathbf{f} - \Phi\nabla)\, \mathbf{w} dv, \tag{11}$$

for any $\mathbf{w} \in \mathcal{S}^2$.
Then, the (symmetric) interior penalty discontinuous Galerkin method of the problem (2) – (3), reads as folllows:

$$\text{Find } \mathbf{u}_{DG} \in \mathcal{S}_1 \text{ such that } \; B_{sg}(\mathbf{u}_{DG}, \mathbf{w}) = L_{sg}(\mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{S}_1. \tag{12}$$

We notice that this formulation is incosistent for trial and test functions belonging either to the solution space $\mathcal{S}^2$ or to the solution space $H_0^2(\Omega)^2$.
In practice, the right-hand side is approximated by the L^2-projection of the source of the function $\mathbf{f}$ onto the finite element space $\mathcal{S}_1$. We denote the L^2-projection of $\mathbf{f}$ onto $\mathcal{S}_1$ by $\Pi\mathbf{f}$.
For any function $\mathbf{u} \in \mathcal{S}^2$, the energy seminorm, $|||\cdot|||_{sg}$, associated with the bilinear form, $B_{sg}(\cdot,\cdot)$, is defined by

$$\begin{aligned} |||\mathbf{u}|||_{sg} \;\; = \;\; & \Big(||\{(\lambda+\mu)g^2\}^{1/2} D_h^2 \mathbf{u}||_{\Omega}^2 + ||(\lambda+\mu)^{1/2}\nabla_h \mathbf{u}||_{\Omega}^2 \\ & + ||(\mu g^2)^{1/2}\Delta_h \mathbf{u}||_{\Omega}^2 + ||\mu^{1/2}\nabla_h \mathbf{u}||_{\Omega}^2 \\ & + ||\gamma^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\xi^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 \\ & + ||\alpha^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\beta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\delta^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 \Big)^{1/2}. \end{aligned} \tag{13}$$

Notice that $[\![\nabla\mathbf{u}]\!] \equiv [\![\nabla\mathbf{u}\cdot n]\!]$ and $\langle\nabla\mathbf{u}\rangle \equiv \langle\nabla\mathbf{u}\cdot n\rangle$.

5.1 Stability Bounds of Lifting Operators

In this section, our main concern is to derive the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$.
Lemma 5.1.1. *Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ be the trace liftings defined in* (6), *in* (7), *in* (8) *as well as in* (9), *respectively. Then, for $\mathbf{u} \in \mathcal{S}^2$, the following bounds hold:*

$$||\mathcal{L}_1(\mathbf{u})||_{\Omega}^2 \leqslant C_1(\lambda,\mu,g^2)\left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right), \tag{14}$$

$$||\mathcal{L}_2(\mathbf{u})||_{\Omega}^2 \leqslant C_2(\lambda,\mu)||\xi_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2, \tag{15}$$

$$||\mathcal{L}_3(\mathbf{u})||_{\Omega}^2 \leqslant C_3(\mu,g^2)\left(||\alpha_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\beta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right), \tag{16}$$

$$||\mathcal{L}_4(\mathbf{u})||_{\Omega}^2 \leqslant C_4(\mu)||\delta_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2, \tag{17}$$

where

$$C_1(\lambda,\mu,g^2) = \frac{1}{(\lambda+\mu)g^2},\; C_2(\lambda,\mu) = \frac{1}{\lambda+\mu},\; C_3(\mu,g^2) = \frac{1}{\mu g^2},\; C_4(\mu) = \frac{1}{\mu} \tag{18}$$

are positive constants, that are independent of $\mathbf{u}$ and of discretization parameters. We denote by $\gamma_1 : \Gamma_0 \to \Re$, $\zeta_1 : \Gamma_0 \to \Re$, $\xi_1 : \Gamma_0 \to \Re$, $\alpha_1 : \Gamma_0 \to \Re$, $\beta_1 : \Gamma_0 \to \Re$ and $\delta_1 : \Gamma_0 \to \Re$ piecewise constant functions, defined by

$$\gamma_1 = C_{\gamma_1}(\lambda+\mu)g^2\left\langle\frac{\mathbf{p^6}}{\mathbf{h^3}}\right\rangle, \quad \zeta_1 = C_{\zeta_1}(\lambda+\mu)g^2\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle, \quad \xi_1 = C_{\xi_1}(\lambda+\mu)\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle,$$

$$\alpha_1 = C_{\alpha_1}\mu g^2\left\langle\frac{\mathbf{p^6}}{\mathbf{h^3}}\right\rangle, \quad \beta_1 = C_{\beta_1}\mu g^2\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle, \quad \delta_1 = C_{\delta_1}\mu\left\langle\frac{\mathbf{p^2}}{\mathbf{h}}\right\rangle,$$

with C_{γ_1}, C_{ζ_1}, C_{ξ_1}, C_{α_1}, C_{β_1} as well as C_{δ_1} sufficiently large positive constants depending only on the mesh parameters.

Proof. We denote by $\Pi : L^2(\Omega)^2 \to \mathcal{S}_1$ the (orthogonal) L^2-projection operator onto the finite element $\mathcal{S}_1$. By invoking the definition of the L^2-norm, the orthogonality of the L^2-projection operator, the definition of the trace lifting $\mathcal{L}_1$ and the Cauchy-Schwarz inequality, we consequently get

$$\begin{aligned} ||\mathcal{L}_1(\mathbf{u})||_{\Omega} \;\; \leqslant \;\; & \sup_{\mathbf{z}\in L^2(\Omega)^2} \frac{\left(||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||_{\Gamma_0}^2 + ||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||_{\Gamma_0}^2\right)^{\frac{1}{2}}}{||\mathbf{z}||_{\Omega}} \\ & \times \left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{19}$$

Thereby, to complete the proof, it only remains to estimate each of the mean value terms appearing on the right-hand side of (19). Hence, by using the shape regularity, the mesh regularity, the bounded local variation of the polynomial degree distribution assumptions on the finite element space $\mathcal{S}_1$, as well as the inverse inequality, we deduce

$$||\gamma_1^{-1/2}\langle\nabla(\Pi\mathbf{z})\rangle||^2_{\Gamma_0} \leqslant \frac{1}{2(\lambda+\mu)g^2}||\mathbf{z}||^2_{\Omega} \quad and \quad ||\zeta_1^{-1/2}\langle\Pi\mathbf{z}\rangle||^2_{\Gamma_0} \leqslant \frac{1}{2(\lambda+\mu)g^2}||\mathbf{z}||^2_{\Omega}. \tag{20}$$

To boot, insertion of the inequalities (20) on the right-hand side of (19) yields

$$||\mathcal{L}_1(\mathbf{u})||^2_{\Omega} \leqslant \frac{1}{(\lambda+\mu)g^2}\left(||\gamma_1^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0} + ||\zeta_1^{1/2}[\![\nabla\mathbf{u}]\!]||^2_{\Gamma_0}\right),$$

which is one of the desired results. What is more, by following the above procedure step by step, we shall bound the trace lifitng $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$, respectively. □

5.2 Coercivity of Bilinear Form

In this section, our goal is to examine the coercivity of the bilinear form $B_{sg}(\cdot,\cdot)$ for the symmetric interior penalty discontinuous Galerkin finite element method.

Proposition 5.2.1. *Let $\gamma:\Gamma_0\to\Re$, $\zeta:\Gamma_0\to\Re$, $\xi:\Gamma_0\to\Re$, $\alpha:\Gamma_0\to\Re$, $\beta:\Gamma_0\to\Re$ and $\delta:\Gamma_0\to\Re$ be piecewise constant functions, such that $\gamma>2\gamma_1$, $\zeta>2\zeta_1$, $\xi>2\xi_1$, $\alpha>2\alpha_1$, $\beta>2\beta_1$ as well as $\delta>2\delta_1$. Then, the bilinear form $B_{sg}(\cdot,\cdot)$, defined in* (10)*, is coercive in the sense that*

$$B_{sg}(\mathbf{u},\mathbf{u}) \geqslant m|||\mathbf{u}|||^2_{sg} \quad \forall\mathbf{u}\in\mathcal{S}_1, \tag{21}$$

where m is a positive constant depending only on the mesh parameters.

Proof. Substituting $\mathbf{u}$ for $\mathbf{w}$ in the bilinear form (10), then by applying the Cauchy-Schwarz inequality, the Young inequality and next by invoking the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and by using the mathematical inequalities (14) – (17), it is clear that

$$\begin{aligned} B_{sg}(\mathbf{u},\mathbf{u}) \;\geqslant\; & \frac{1}{2}||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{u}||^2_{\Omega} + \frac{1}{2}||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||^2_{\Omega} \\ & + \frac{1}{2}||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||^2_{\Omega} + \frac{1}{2}||\mu^{1/2}\nabla_h\mathbf{u}||^2_{\Omega} \\ & + ||(\gamma-2\gamma_1)^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0} + ||(\zeta-2\zeta_1)^{1/2}[\![\nabla\mathbf{u}]\!]||^2_{\Gamma_0} \\ & + ||(\xi-2\xi_1)^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0} + ||(\alpha-2\alpha_1)^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0} \\ & + ||(\beta-2\beta_1)^{1/2}[\![\nabla\mathbf{u}]\!]||^2_{\Gamma_0} + ||(\delta-2\delta_1)^{1/2}[\![\mathbf{u}]\!]||^2_{\Gamma_0}. \end{aligned} \tag{22}$$

Since we assumed $\gamma>2\gamma_1$, $\zeta>2\zeta_1$, $\xi>2\xi_1$, $\alpha>2\alpha_1$, $\beta>2\beta_1$ and $\delta>2\delta_1$, coercivity follows, i.e.

$$B_{sg}(\mathbf{u},\mathbf{u}) \geqslant m|||\mathbf{u}|||^2_{sg},$$

which is the desired result. We denote by the constant m the minimum of the coefficients on the right-hand side of (22). □

5.3 Continuity of Bilinear Form

With the definition of the energy seminorm, (13), we have the following continuity result for the bilinear form $B_{sg}(\cdot,\cdot)$, based on the Cauchy-Schwarz inequality.

Proposition 5.3.1. *Let $\gamma:\Gamma_0\to\Re$, $\zeta:\Gamma_0\to\Re$, $\xi:\Gamma_0\to\Re$, $\alpha:\Gamma_0\to\Re$, $\beta:\Gamma_0\to\Re$ and $\delta:\Gamma_0\to\Re$ be piecewise constant functions, such that $\gamma>2\gamma_1$, $\zeta>2\zeta_1$, $\xi>2\xi_1$, $\alpha>2\alpha_1$, $\beta>2\beta_1$ as well as $\delta>2\delta_1$. Then, the bilinear form $B_{sg}(\cdot,\cdot)$, defined in* (10)*, is continuous in the sense that*

$$B_{sg}(\mathbf{u},\mathbf{w}) \leqslant C|||\mathbf{u}|||_{sg}|||\mathbf{w}|||_{sg} \quad \forall\mathbf{u},\mathbf{w}\in\mathcal{S}^2, \tag{23}$$

where C is a positive constant depending only on the mesh parameters.

Proof. Let $\mathbf{u}, \mathbf{w} \in \mathcal{S}^2$, we can obtain (23) by applying at first the triangle inequality in the bilinear form, then the Cauchy-Schwarz inequality and finally by recalling the stability of the trace liftings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and therefore employing the mathematical expressions (14) – (17). For that reason, we deduce

$$\begin{aligned} B_{sg}(\mathbf{u},\mathbf{w}) \quad \leqslant \quad & \Big(2||\{(\lambda+\mu)g^2\}^{1/2} D_h^2\mathbf{u}||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla_h\mathbf{u}||_\Omega^2 \\ & + 2||(\mu g^2)^{1/2}\Delta_h\mathbf{u}||_\Omega^2 + 2||\mu^{1/2}\nabla_h\mathbf{u}||_\Omega^2 \\ & + \frac{1}{2}||\gamma^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\zeta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\xi^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 \\ & + \frac{1}{2}||\alpha^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\beta^{1/2}[\![\nabla\mathbf{u}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\delta^{1/2}[\![\mathbf{u}]\!]||_{\Gamma_0}^2\Big)^{1/2} \\ & \times \Big(2||\{(\lambda+\mu)g^2\}^{1/2} D_h^2\mathbf{w}||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla_h\mathbf{w}||_\Omega^2 \\ & + 2||(\mu g^2)^{1/2}\Delta_h\mathbf{w}||_\Omega^2 + 2||\mu^{1/2}\nabla_h\mathbf{w}||_\Omega^2 \\ & + \frac{1}{2}||\gamma^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\zeta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\xi^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 \\ & + \frac{1}{2}||\alpha^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\beta^{1/2}[\![\nabla\mathbf{w}]\!]||_{\Gamma_0}^2 + \frac{1}{2}||\delta^{1/2}[\![\mathbf{w}]\!]||_{\Gamma_0}^2\Big)^{1/2}. \end{aligned} \tag{24}$$

Also, by the use of definition of energy seminorm, (13), on the right-hand side of (24), we reach the desired result. □

6 A POSTERIORI ERROR ANALYSIS

In this section, we want to conduct an error analysis for IPDG finite element method (12).

6.1 Finite Element Spaces

In this section, we wish to modify a little the finite element space, defined in section 4, so that it can include either triangular or quadrilateral elements.
Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint triangular or quadrilateral elements $K \in \mathcal{T}$. We assume that the elemental edges are straight line segments.
For a non-negative integer p, we denote by $\mathcal{P}_p(\hat{K})$ the set of all polynomials of total degree at most p if $\hat{K}$ is either the reference triangle or the set of all tensor product polynomials on $\hat{K}$ of degree at most p in each coordinate direction if $\hat{K}$ is the reference quadrilateral. For $p \geqslant 2$ we consider the finite element space

$$\mathcal{S}_1 \equiv [\mathcal{S}_h^p]^2 := \left\{\mathbf{u} \in L^2(\Omega)^2 : \mathbf{u}|_K \circ F_K \in \mathcal{P}_p(\hat{K})^2,\ K \in \mathcal{T}\right\}. \tag{25}$$

We collect the h_K into the elementwise constant function

$$\mathbf{h} : \Omega \to \Re, \text{ with } \mathbf{h}|_K = h_K, K \in \mathcal{T} \text{ and } \mathbf{h}|_e = \langle\mathbf{h}\rangle, e \subset \Gamma_0.$$

We shall assume throughout that the families of meshes considered are locally quasiuniform or in other words the mesh size function $\mathbf{h}$ has bounded local variation.
Then, the piecewise constant stabilization parameters $\gamma : \Gamma_0 \to \Re$, $\zeta : \Gamma_0 \to \Re$, $\xi : \Gamma_0 \to \Re$, $\alpha : \Gamma_0 \to \Re$, $\beta : \Gamma_0 \to \Re$ and $\delta : \Gamma_0 \to \Re$ are defined by

$$\gamma = C_\gamma(\lambda+\mu)g^2(\mathbf{h}|_e)^{-3}, \quad \zeta = C_\zeta(\lambda+\mu)g^2(\mathbf{h}|_e)^{-1}, \quad \xi = C_\xi(\lambda+\mu)(\mathbf{h}|_e)^{-1}, \tag{26}$$

$$\alpha = C_\alpha\mu g^2(\mathbf{h}|_e)^{-3}, \quad \beta = C_\beta\mu g^2(\mathbf{h}|_e)^{-1}, \quad \delta = C_\delta\mu(\mathbf{h}|_e)^{-1}, \tag{27}$$

with C_γ, C_ζ, C_ξ, C_α, C_β as well as C_δ sufficiently large positive constants.

6.2 Recovery Operator

The use of a recovery operator, mapping elements of $\mathcal{S}_1$ onto a C^1-conforming space consisting of macro-elements of degree $p+2$, is a significant tool helping us conduct a posteriori error analysis. The family of macro-elements

considered will be higher-order versions of the classical Hsieh-Clough-Tocher macro-element, constructed in [7]. This mapping is constructed via averages of the nodal basis functions (see [14, 3, 13, 12]).
The corresponding finite element space consisting of the above macro-elements will be denoted by $\mathcal{S}_2 \equiv [\tilde{\mathcal{S}}_h^m]^2$.
Let us consider the standard Lagrange basis for a polynomial of degree p, where $p \geqslant 2$. A crucial observation here is that the set of the nodal points of the Lagrange basis is a subset of the set of the nodal points of the macro-elements of degree $p+2$. In that case, the corresponding finite element space is $\mathcal{S}_2 \equiv \left[\tilde{\mathcal{S}}_h^{p+2}\right]^2$ and it will be used in the following Lemma.

Lemma 6.2.1. *Let us assume that the mesh $\mathcal{T}$ is constructed as in section 6.1. Then, there exists an operator $\mathbf{E}_{op} : \mathcal{S}_1 \to \mathcal{S}_2 \cap H_0^2(\Omega)^2$ satisfying the following error bounds:*

$$\sum_{k\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)|_{j,K}^2 \leqslant C_1 \left(||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + ||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 \right), \tag{28}$$

with $j = 2$ and

$$\sum_{k\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)|_{j,K}^2 \leqslant C_2 ||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2, \tag{29}$$

with $j = 1$. We denote by $C_1, C_2 > 0$ some constants that are independent of $\mathbf{h}$ and $\mathbf{u}_h$.

Proof. For each nodal point np of the C^1-conforming finite element space $\mathcal{S}_2$, we define ω_{np} to be the set of $K \in \mathcal{T}$ that share the nodal point np, i.e.,

$$\omega_{np} := \{K \in \mathcal{T} : np \in \mathcal{T}\}.$$

Furthermore, $|\omega_{np}|$ will denote the cardinality of ω_{np}. Next, we define the operator $\mathbf{E}_{op} : \mathcal{S}_1 \to \mathcal{S}_2 \cap H_0^2(\Omega)^2$ by

$$N_{np}(\mathbf{E}_{op}(\mathbf{u}_h)) = \begin{cases} \frac{1}{|\omega_{np}|} \sum_{K\in\omega_{np}} N_{np}(\mathbf{u}_h|_K), & \text{if } np \notin \Gamma_c \\ 0, & \text{if } np \in \Gamma_c, \end{cases} \tag{30}$$

where N_{np} is any nodal variable at np and np is any nodal point of $\mathcal{S}_2$. Note that

$$N_{np}(\mathbf{E}_{op}(\mathbf{u}_h)) = N_{np}(\mathbf{u}_h), \quad \text{if } np \in \text{int}K.$$

We denote by $\mathcal{N}$ the set of all nodal variables of $\mathcal{S}_2$ defined on every element of $\mathcal{T}$, i.e., they may be discontinuous across element boundaries. Then, we can split $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1,$$

where $\mathcal{N}_0$ and $\mathcal{N}_1$ consisting of the nodal variables corresponding to the function evaluations and those involving partial and normal derivatives of the function, respectively.
The use of an inverse estimate yields

$$\sum_{K\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(u_h)|_{j,K}^2 \leqslant C \left|\left|\mathbf{h}^{-j}(\mathbf{u}_h - \mathbf{E}_{op}(u_h))\right|\right|_{\Omega}^2, \tag{31}$$

with C a positive constant which is independent of $\mathbf{h}$ and $\mathbf{u}_h$. After that, the equivalence of norms in a finite-dimensional vector space along with a scaling argument gives

$$\left|\left|\mathbf{h}^{-j}(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h))\right|\right|_{\Omega}^2 \leqslant C \sum_{i=0}^{1} \sum_{N_{np}\in\mathcal{N}_i : np\in K} h_K^{2(i+1-j)} \left(N_{np}\left(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2. \tag{32}$$

Now, by recalling the arithmetic-geometric mean inequality, we conclude that

$$\sum_{N_{np}\in\mathcal{N}_0 : np\in K} h_K^{2(1-j)} \left(N_{np}\left(\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)\right)\right)^2 \leqslant C||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2. \tag{33}$$

What is more, it's time for us to turn to the nodal variables in $\mathcal{N}_1$. We further split $\mathcal{N}_1$ into

$$\mathcal{N}_1 = \mathcal{N}_1^n \cup \mathcal{N}_1^p,$$

where $\mathcal{N}_1^n$ is the set of the nodal variables of normal derivatives across element edges and $\mathcal{N}_1^p$ is the set of nodal variables representing partial derivatives on elemental vertices. Hence, we shall follow arguments in a same way for $\mathcal{N}_1^n$ and $\mathcal{N}_1^p$ as in (33).
Ergo, we reach the conclusion that

$$\sum_{K\in\mathcal{T}} |\mathbf{u}_h - \mathbf{E}_{op}(\mathbf{u}_h)|_{j,K}^2 \leqslant C \left(||\mathbf{h}^{1/2-j}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + ||\mathbf{h}^{3/2-j}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 \right),$$

being the desired result for $j = 2$. We analogously prove the inequality (29). □

6.3 A Posteriori Error Estimates

In this section, overall our research endeavor focuses mainly on establishing a reliable a posteriori error estimate of residual type for the (symmetric) interior penalty discontinuous Galerkin method in the corresponding energy seminorm, when the analytical solution $\mathbf{u}$ of (2) – (3) satisfies $\mathbf{u} \in H_0^2(\Omega)^2$.

Theorem 6.3.1. *Let* $\mathbf{u} \in H_0^2(\Omega)^2$ *be the solution to* (2) – (3)*,* $\mathbf{u}_h \in \mathcal{S}_1$ *be the approximate solution obtained by the interior penalty discontinuous Galerkin method and* γ*,* ζ*,* ξ*,* α*,* β *together with* δ *as in* (26) – (27)*. Then, there exists a positive constant* C*, independent of* $\mathbf{h}$*,* $\mathbf{u}$ *and* $\mathbf{u}_h$*, so that*

$$\begin{aligned}
&|||\mathbf{u}-\mathbf{u}_h|||_{sg}^2\\
&\leqslant C\Big(||\mathbf{h}^2\{\tilde{\mathbf{f}}-(\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h+(\lambda+\mu)D_h^2\mathbf{u}_h-\mu g^2\Delta_h^2\mathbf{u}_h+\mu\Delta_h\mathbf{u}_h\}||_\Omega^2\\
&+C_p^2\Big\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2+(\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&+(\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2+\mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\Big\}\\
&+(\lambda+\mu)g^2||\mathbf{h}^{3/2}[\![\nabla D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2+(\lambda+\mu)g^2||\mathbf{h}^{1/2}[\![D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\\
&+(\lambda+\mu)||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2+\mu g^2||\mathbf{h}^{3/2}[\![\nabla\Delta\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\\
&+\mu g^2||\mathbf{h}^{1/2}[\![\Delta\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2+\mu||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\Big),
\end{aligned} \tag{34}$$

where $C_p := \max\{C_\gamma, C_\zeta, C_\xi, C_\alpha, C_\beta, C_\delta\}$ *and* $\tilde{\mathbf{f}} = \mathbf{f} - \Phi\nabla$*.*

Proof. Let $\mathbf{w}_h \in \mathcal{S}_1$, $\mathbf{w} \in H_0^2(\Omega)^2$, $\eta = \mathbf{w} - \mathbf{w}_h$ and $\mathbf{E}_{op}(\mathbf{u}_h) \in \mathcal{S}_2 \cap H_0^2(\Omega)^2$ be as in Lemma 6.2.1. We shall use this notation with intention to decompose the error as follows:

$$\mathbf{e} := \mathbf{u}-\mathbf{u}_h = (\mathbf{u}-\mathbf{E}_{op}(\mathbf{u}_h)) + (\mathbf{E}_{op}(\mathbf{u}_h)-\mathbf{u}_h) \equiv \mathbf{e}^c + \mathbf{e}^d. \tag{35}$$

Since $\mathbf{u}$ is the solution to the weak problem, we get

$$B_{sg}(\mathbf{u},\mathbf{w}) = L_{sg}(\mathbf{w}) \quad \text{as} \quad \mathcal{L}_i(\mathbf{u}) = \mathcal{L}_i(\mathbf{w}) = 0 \quad \forall i = 1,2,3,4.$$

As a consequence,

$$B_{sg}(\mathbf{e},\mathbf{w}) = B_{sg}(\mathbf{u},\mathbf{w}) - B_{sg}(\mathbf{u}_h,\mathbf{w}) = L_{sg}(\eta) - B_{sg}(\mathbf{u}_h,\eta) \tag{36}$$

and it also holds that

$$B_{sg}(\mathbf{e},\mathbf{w}) = B_{sg}(\mathbf{e}^c,\mathbf{w}) + B_{sg}(\mathbf{e}^d,\mathbf{w}). \tag{37}$$

Thereby, (36) and (37) entail that

$$B_{sg}(\mathbf{e}^c,\mathbf{w}) = L_{sg}(\eta) - B_{sg}(\mathbf{u}_h,\eta) - B_{sg}(\mathbf{e}^d,\mathbf{w}). \tag{38}$$

After that, by employing the decomposition of the error, (35), we obtain for the energy seminorm of the error $|||\mathbf{u}-\mathbf{u}_h|||_{sg}$

$$\begin{aligned}
|||\mathbf{u}-\mathbf{u}_h|||_{sg}^2 \leqslant\ & 2\Big(||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2+||\{(\lambda+\mu)g^2\}^{1/2}D_h^2\mathbf{e}^d||_\Omega^2\\
&+||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2+||(\lambda+\mu)^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\\
&+||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2+||(\mu g^2)^{1/2}\Delta_h\mathbf{e}^d||_\Omega^2\\
&+||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2+||\mu^{1/2}\nabla_h\mathbf{e}^d||_\Omega^2\Big)\\
&+C_p\Big\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&+(\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2+(\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&+\mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\\
&+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\Big\}.
\end{aligned} \tag{39}$$

Thus, to complete the proof, it only remains to estimate the terms of $\mathbf{e}^c$ and $\mathbf{e}^d$, enclosed into the parenthesis on the right-hand side of (39).

For the terms of $\mathbf{e}^d$, by recalling the Lemma 6.2.1 on the right-hand side of (39), we reach the conclusion for the energy seminorm of the error

$$\begin{aligned} |||\mathbf{u}-\mathbf{u}_h|||_{sg}^2 \leqslant\ & 2\left(||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2 \right.\\ & \left. +||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2 + ||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2\right)\\ & + C_p\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right.\\ & + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\ & + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\\ & \left.+\mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\}. \end{aligned} \tag{40}$$

Now, it only remains to estimate the terms of $\mathbf{e}^c$.
Next, we notice that $\mathcal{L}_i(\mathbf{e}^c)=0$, $i=1,2,3,4$, since $\mathbf{e}^c\in H_0^2(\Omega)^2$. Ergo, upon setting $\mathbf{w}=\mathbf{e}^c$ in (38), we deduce

$$B_{sg}(\mathbf{e}^c,\mathbf{e}^c) = L_{sg}(\eta) - B_{sg}(\mathbf{u}_h,\eta) - B_{sg}(\mathbf{e}^d,\mathbf{e}^c). \tag{41}$$

It therefore derives

$$B_{sg}(\mathbf{e}^c,\mathbf{e}^c) \leqslant |L_{sg}(\eta) - B_{sg}(\mathbf{u}_h,\eta)| + |B_{sg}(\mathbf{e}^d,\mathbf{e}^c)|, \tag{42}$$

where

$$B_{sg}(\mathbf{e}^c,\mathbf{e}^c) = ||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2 + ||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2 + ||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2. \tag{43}$$

So, by applying at first the Cauchy-Schwarz inequality in the bilinear form (10) and then by invoking the stability of lifting operators, (14) – (17), we conclude

$$\begin{aligned} |B_{sg}(\mathbf{e}^d,\mathbf{e}^c)| \leqslant\ & C_p^{1/2}\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right.\\ & + (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\ & \left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\}^{1/2}\\ & \times\left(2||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2 + 2||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega\right.\\ & \left.+2||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2 + 2||\mu^{1/2}\Delta\mathbf{e}^c||_\Omega^2\right)^{1/2}. \end{aligned} \tag{44}$$

To proceed, thanks to the fact that $\mathbf{u}_h,\mathbf{w}_h\in\mathcal{S}_1$ and $\mathbf{w}\in H_0^2(\Omega)^2$, we can use the definitions of the lifting operators, (6) – (9).
Fix $\mathbf{w}_h$ to be the elementwise linear approximation to $\mathbf{e}^c$ such that

$$|\mathbf{e}^c-\mathbf{w}_h|_{j,K} \leqslant Ch_K^{m-j}|\mathbf{e}^c|_{m,K} \tag{45}$$

for $C>0$, independent of $\mathcal{T}$, for $0\leqslant j\leqslant m\leqslant 2$ and $K\in\mathcal{T}$ (see [6]).
At this point, using the mathematical inequality, (45), the Cauchy-Schwarz inequality and recalling subsequently the stability of lifting operators, (14) – (17), we can estimate the first two terms on the right-hand side of (42). Afterwards, by combining the deriving inequality with the mathematical expressions (42), (43), (44), we arrive to the conclusion

$$\begin{aligned} & ||\{(\lambda+\mu)g^2\}^{1/2}D^2\mathbf{e}^c||_\Omega^2 + ||(\lambda+\mu)^{1/2}\nabla\mathbf{e}^c||_\Omega^2 + ||(\mu g^2)^{1/2}\Delta\mathbf{e}^c||_\Omega^2 + ||\mu^{1/2}\nabla\mathbf{e}^c||_\Omega^2\\ & \leqslant C\left(||\mathbf{h}^2\{\tilde{\mathbf{f}} - (\lambda+\mu)g^2\left(D_h^2\right)^2\mathbf{u}_h + (\lambda+\mu)D_h^2\mathbf{u}_h - \mu g^2\Delta_h^2\mathbf{u}_h + \mu\Delta_h\mathbf{u}_h\}||_\Omega^2\right.\\ & + C_p^2\left\{(\lambda+\mu)g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + (\lambda+\mu)g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2\right.\\ & + (\lambda+\mu)||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu g^2||\mathbf{h}^{-3/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\\ & \left.+\mu g^2||\mathbf{h}^{-1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_0}^2 + \mu||\mathbf{h}^{-1/2}[\![\mathbf{u}_h]\!]||_{\Gamma_0}^2\right\}\\ & + (\lambda+\mu)g^2||\mathbf{h}^{3/2}[\![\nabla D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + (\lambda+\mu)g^2||\mathbf{h}^{1/2}[\![D^2\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\\ & + (\lambda+\mu)||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + \mu g^2||\mathbf{h}^{3/2}[\![\nabla\Delta\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\\ & \left.+\mu g^2||\mathbf{h}^{1/2}[\![\Delta\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2 + \mu||\mathbf{h}^{1/2}[\![\nabla\mathbf{u}_h]\!]||_{\Gamma_{\text{int}}}^2\right). \end{aligned} \tag{46}$$

Finally, insertion of inequality (46) on the right-hand side of (40) gives the desired result. □

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