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LIMIT LOAD AND DEFORMATION ANALYSIS OF STEEL FRAMES WITH NONLINEAR BEHAVIOR AND MULTI-COMPONENT INTERACTION

Marina-Myrto S. Manola, Vlasis K. Koumousis.

Institute of Structural Analysis & Aseismic Research
National Technical University of Athens
NTUA, Zographou Campus GR-15780, Athens, Greece
e-mail: m.m.manola@gmail.com, vkoum@central.ntua.gr

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Abstract. *This work deals with limit load and deformation analysis of plane steel structures following nonlinear hardening/softening behavior under the effect of combined stresses. Nonlinear yield criteria accounting for axial force-bending moment and axial-shear force-bending moment interaction are adopted, while cross-sectional yielding is considered isotropic. The ultimate structural state is determined by solving an optimization problem with linear equilibrium, compatibility and yield constraints together with a nonlinear complementarity condition. Due to the disjunctive nature of the latter, the problem is stated as nonlinear programming problem following a penalty function formulation. The yield condition and nonlinear hardening/softening are expressed herein utilizing a local linearization technique which does not affect the size nor the linearity of the yield constraint formulation. Numerical results for plane steel frames are presented that verify the validity of the proposed method and demonstrate the effect of multi-component interaction on structural ultimate state.*

1 INTRODUCTION

Limit and deformation analysis of plane structures is dealt herein in the framework of mathematical programming. This formulation, pioneered by Maier et al. [1-4], is based on the a priori piecewise linearization of yield condition and constitutive relations under holonomic or nonholonomic assumption. The ultimate structural state is assessed through an optimization problem that aims at the maximization of the load factor subjected to equilibrium, yield and deformation i.e. compatibility constraints.

A variety of alternative mathematical programming procedures such as iterative Linear Programming, Quadratic Programming, Restricted Basis Linear Programming, Parametric Linear Complementarity and Parametric Quadratic Programming procedures have been applied for limit analysis of structures [5,6]. The recent development of mathematical programming algorithms appropriate for Mathematical Programming with Equilibrium Constraints (MPEC) problems [7] has extended the potential of the proposed methods for structural analysis for both holonomic and nonholonomic assumptions [8-13].

The aim of this work is to address limit load and deformation analysis of structures considering nonlinear interaction and structural behavior. The existing formulation is based on the a priori linearization of the yield surface and constitutive laws. Herein, a new approach is proposed that retains the nonlinearity of the yield surface applying a local linearization technique. Moreover, isotropic nonlinear hardening/softening cross-sectional behavior is efficiently incorporated. The ultimate state of the structure is determined as an optimization problem with linear equilibrium, compatibility and yield constraints together with a nonlinear complementarity constraint. The disjunctive nature of the latter enforces the formulation of a non-linear programming problem using a penalty function method.

The organization of the paper is as follows. First, the governing relations of holonomic elastoplastic problem based on equilibrium, kinematical and constitutive relations are summarized. Then, the formulation of limit and deformation analysis as a MPEC problem is presented incorporating the axial-bending moment (NM) and axial-shear force-bending moment (NQM) interaction. Subsequently a numerical example of a steel frame is presented that illustrates the applicability of the proposed method.

2 PROBLEM FORMULATION

2.1 Basic Assumptions

The entire formulation is based on the following assumptions. Plane frames consist of n_{el} straight prismatic elements, with n_f nodal degrees of freedom, subjected only to nodal loading for reasons of simplicity. Frame displacements are assumed small enough so that the equilibrium equations refer to the initial undeformed configuration. Plastic hinges are considered formed only at critical sections, i.e. the end sections of the elements, whereas the remaining parts behave elastically. Nonlinear yield criteria accounting for axial force-bending moment (NM) and axial-shear force-bending moment (NQM) interaction are adopted, while the cross-sectional nonlinear inelastic behavior is considered isotropic. Furthermore, under the external loading, if local unloading occurs, is assumed happening along the load displacement path and not as elastic unloading, adopting a holonomic, i.e. path-independent structural behavior. Although this is a simplified assumption, especially for the case of softening behavior, it can be considered reasonable for monotonically increasing external actions [3,12,13].

2.2 Physical Considerations

2.2.1 Equilibrium

Each plane beam element develops six stress resultants at its ends, as shown in Figure 1. Three of them are considered as independent (12,13), while the remaining dependent actions can be evaluated by applying the three equilibrium equations. The independent actions herein are the axial force (s_1^i), bending moment at the start node j (s_2^i) and bending moment at the end node k (s_3^i). The equilibrium for the whole structure is then established in terms of the unknown vector of stresses of all members as:

$$\boldsymbol{B} \cdot \boldsymbol{s} = a \cdot \boldsymbol{f} + \boldsymbol{f}_d \tag{1}$$

where $\boldsymbol{B}$ is the ($n_f \times 3n_{el}$) structural equilibrium matrix, assembled by the corresponding element equilibrium matrices arranged in a block diagonal manner, $\boldsymbol{s}$ is a ($3n_{el} \times 1$) vector of all stresses in local systems, a is a scalar load factor, $\boldsymbol{f}$ the ($n_f \times 1$) vector of nodal loading in the global system and $\boldsymbol{f}_d$ is the ($n_f \times 1$) fixed nodal load vector.

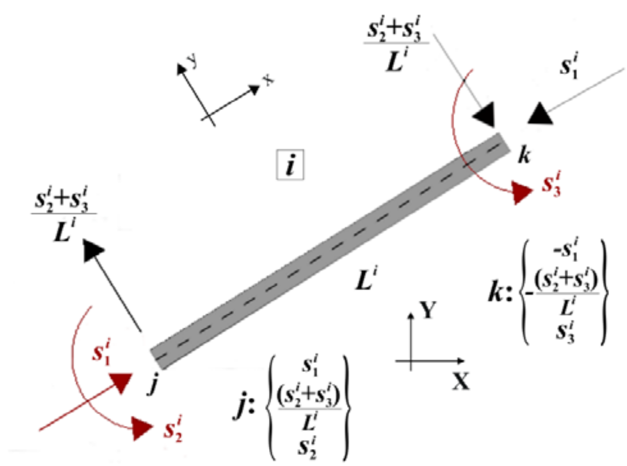

Figure 1.Frame element *i* with equilibrated stress resultants-end actions.

2.2.2 Compatibility Condition

The compatibility condition for the whole structure is then given by the following linear compatibility relation:

$$\boldsymbol{q} = \boldsymbol{B}^T \cdot \boldsymbol{u} \tag{2}$$

where $\boldsymbol{q}$ is the ($3n_{el} \times 1$) deformation vector of the structure and $\boldsymbol{u}$ is the ($n_f \times 1$) nodal displacement vector.

2.2.3 Constitutive Relations

For the entire structure the deformation is decomposed into an elastic and plastic component as:

$$\boldsymbol{q} = \boldsymbol{e} + \boldsymbol{p} = \boldsymbol{S}^{-1} \cdot \boldsymbol{s} + \boldsymbol{N} \cdot \mathbf{z} \tag{3}$$

where $\boldsymbol{e}$ is the ($3n_{el} \times 1$) elastic and $\boldsymbol{p}$ the ($3n_{el} \times 1$) plastic component of deformation respectively, $\boldsymbol{S}$ is the ($3n_{el} \times 3n_{el}$) assembled block diagonal matrix of all element stiffness matrices, $\boldsymbol{N}$ and $\mathbf{z}$ are defined in the following section.

2.2.4 Yield Condition

Yield condition is the one that denotes the limit between elastic and plastic region under the effect of combined stresses. Herein, two considerations are examined, i.e. axial force-bending moment (NM) interaction and axial-shear force-bending moment (NQM) interaction. In general, the nonlinear yield criterion is beforehand appropriately linearized either with linear segments (case of 2D interaction) or plane triangles (case of 3D interaction), offering computational advantages for the constraint formulation of the problem. The herein proposed method applies the linearization of the yield surface locally for every stress point. The proposed process constitutes an extension of the cone identification approach [14,15] and is based on the concept that for every stress vector only one yield hyperplane is targeted or activated. This hyperplane is not a priori defined, but is determined at each optimization iteration for every stress point. First, the intersection point of every stress vector with the nonlinear yield surface is defined. Then, the corresponding tangent hyperplane and its normal vector are determined (Fig. 2). The normalized ratio of the reserves as compared to the stress vector is of interest, normalized appropriately. Thus, for every cross section only one yield condition is formed as:

$$\boldsymbol{w} = -\boldsymbol{N}^T \cdot \boldsymbol{s} + \boldsymbol{r} \geq \mathbf{0} \tag{4}$$

where $\boldsymbol{w} = \left\{\boldsymbol{w}^1 \ldots \boldsymbol{w}^n\right\}^T$ (($2n_{el} \times 1$) vector), $\boldsymbol{N}$ (($3n_{el} \times 2n_{el}$) matrix) is the assembled block diagonal matrix of all $\boldsymbol{N}^i$ matrices and $\boldsymbol{r} = \left\{\boldsymbol{r}^1 \ldots \boldsymbol{r}^n\right\}^T$ is a ($2n_{el} \times 1$) vector, $\boldsymbol{w}^i$ is the (2×1) vector that contains the moment reserves of both element ends, $\boldsymbol{N}^i$ is the (3×2) matrix that contains all scaled normal vectors of the determined yield hyperplanes and $\boldsymbol{r}^i$ is the (2×1) vector of the yield limits expressed in bending moment terms. It is noted that the implementation of the method is depicted for a 2D yield criterion in Fig. 2. The proposed formulation is nevertheless general and can be efficiently applied for *d*-component interaction.

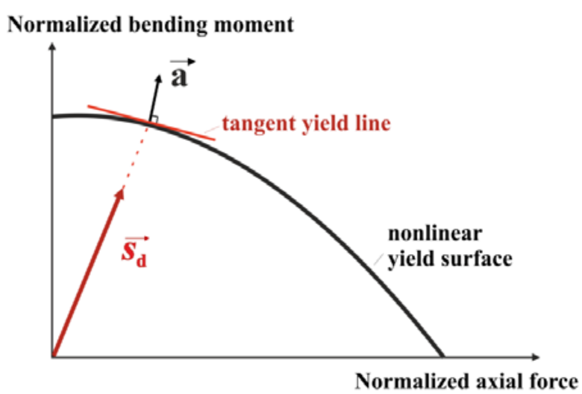

Figure 2: Local linearization of the yield surface for a stress point.

The structural behavior is assumed herein to follow an isotropic nonlinear hardening/softening law. Having experimental data, given as a set of points for axial force-plastic axial deformation and bending moment-plastic rotation, the corresponding nonlinear curves are formed using a curve fitting tool. Then, they are combined by a proportion dictated by the components of the normal vector of the determined yield hyperplane resulting in a curve that relates finally the combined stresses versus plastic rotation. A plastic multiplier z_μ is assigned to every cross section μ at each optimization iteration, the non-zero value of which denotes that the specific cross section has entered the plastic region. Based on the particular non-zero value of the plastic multiplier and having the analytical expression of the nonlinear hardening/softening behavior, the extended/shrunk yield limit of the stress point is directly evaluated (Fig. 3). It is actually the ordinate of the identified curve point that represents the corresponding extended/shrunk yield limit r'_μ required for the formulation of the yield condition. The yield condition for the whole structure is then expressed as:

$$\boldsymbol{w} = -\boldsymbol{N}^T \cdot \boldsymbol{s} + \boldsymbol{r}' \geq \boldsymbol{0} \tag{5}$$

where $\boldsymbol{r}'$ is the $(2n_{el} \times 1)$ vector including the extended limits expressed in terms of bending moment.

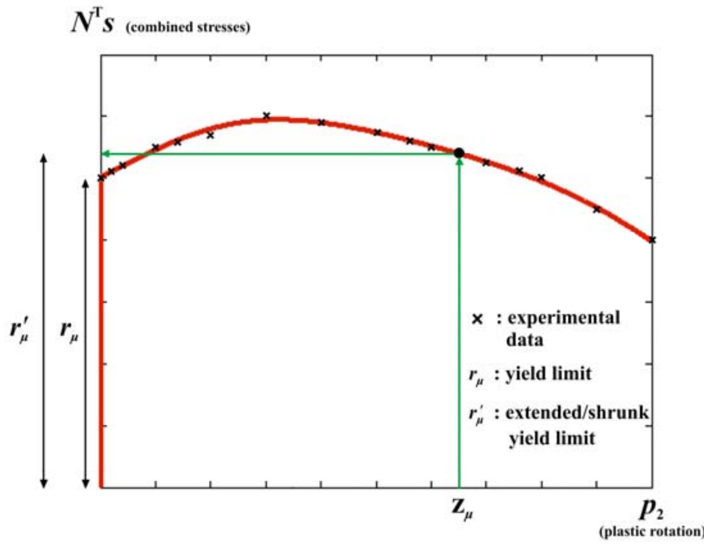

Figure 3: Nonlinear hardening/softening behavior using curve fitting tool.

2.2.5 Complementarity Condition

Complementarity conditions express mutually exclusive situations in the form of an inner product of two nonnegative vectors that should be zero. They indicate that simultaneous activation of plastic deformation and unloading is meaningless. Thus for the entire structure the complementarity constraint is expressed as follows:

$$\boldsymbol{w}^T \cdot \mathbf{z} = \mathbf{0}, \quad \boldsymbol{w} \geq \mathbf{0}, \quad \mathbf{z} \geq \mathbf{0} \tag{6}$$

which holds also component wise.

2.3 Mathematical Formulation

Limit and deformation analysis is treated herein in the framework of mathematical programming. The ultimate load and structural state are determined through an optimization problem that aims at the maximization of the load factor subjected to limitations dictated by equilibrium, compatibility, yield and complementarity condition (eqs (1,2,3,5,6)). Thus the formulation of the optimization problem is given as:

$$\left.\begin{array}{ll} \text{maximize} & a \\ \text{subject to} & \boldsymbol{B}\cdot\boldsymbol{s} = a\cdot\boldsymbol{f} + \boldsymbol{f}_d \\ & \boldsymbol{S}^{-1}\cdot\boldsymbol{s} - \boldsymbol{B}^T\cdot\boldsymbol{u} + \boldsymbol{N}\cdot\mathbf{z} = \mathbf{0} \\ & \boldsymbol{w} = -\boldsymbol{N}^T\cdot\boldsymbol{s} + \boldsymbol{r}' \geq \mathbf{0} \\ & \boldsymbol{w}^T\cdot\mathbf{z} = \mathbf{0} \\ & \mathbf{0} \leq \mathbf{z} \leq \mathbf{z}_u \\ & \boldsymbol{u}_l \leq \boldsymbol{u} \leq \boldsymbol{u}_u \end{array}\right\} \tag{7}$$

The decision variables of the above problem are the stresses $\boldsymbol{s}$, the displacements $\boldsymbol{u}$, the plastic multipliers $\mathbf{z}$ and the load factor a. All constraints are linear except for the complementarity condition, the presence of which converts the problem into a non-convex one. This mathematical expression of the problem constitutes a Mathematical Programming with Equilibrium Constraints (MPEC) problem [7]. The complementarity constraint acts as a multi-switch and is difficult to handle numerically, leading to numerical instabilities. Despite all these inherent difficulties, the MPEC problem (7) can be solved by converting it into a standard, though still nonconvex, nonlinear programming (NLP) problem by suitably treating the complementarity condition. Several techniques have been proposed such as penalty function formulation, relaxation method, active set identification approach, sequential quadratic programming (SQP) and interior point methods, among others [16]. Herein, the penalty function approach is followed [12]. According to this, the complementarity constraint is handled in the objective function by a parametric reformulation, in which an increased value of the parameter ρ exerts a pressure on the complementarity condition leading it to vanish. This formulation is as follows:

$$\left.\begin{array}{ll} \text{maximize} & a - \rho\cdot\boldsymbol{w}^T\cdot\mathbf{z} \\ \text{subject to} & \boldsymbol{B}\cdot\boldsymbol{s} = a\cdot\boldsymbol{f} + \boldsymbol{f}_d \\ & \boldsymbol{S}^{-1}\cdot\boldsymbol{s} - \boldsymbol{B}^T\cdot\boldsymbol{u} + \boldsymbol{N}\cdot\mathbf{z} = \mathbf{0} \\ & \boldsymbol{w} = -\boldsymbol{N}^T\cdot\boldsymbol{s} + \boldsymbol{r}' \geq \mathbf{0} \\ & \mathbf{0} \leq \mathbf{z} \leq \mathbf{z}_u \\ & \boldsymbol{u}_l \leq \boldsymbol{u} \leq \boldsymbol{u}_u \end{array}\right\} \tag{8}$$

The above problem formulation incorporates the local linearization technique of the yield condition at each iteration of the optimization problem. Moreover, the nonlinear hardening/softening structural behavior is incorporated efficiently, affecting neither the linearity nor the size of the yield condition. Matrix $\boldsymbol{N}$ and vector $\boldsymbol{r}'$ are updated for every iteration depending on the determined yield hyperplane and the particular value of the extended/shrunk yield limit of every critical section. Moreover, it is worth noting that this NLP problem is sensitive to the initial values of ρ and its subsequent increase, as well as to the initial values of variables and their lower and upper bounds.

3 NUMERICAL EXAMPLE

The optimization problem incorporating the proposed method is implemented in Matlab code using *fmincon* solver (appropriate for the minimization of constrained nonlinear multivariable function), with the interior-point algorithm selected as optimization method. The aim is to verify the applicability of the introduced approach, validate its efficiency and compare the analysis results with existing ones for the following cases:

- Case (a): **NM** interaction with :
    1. Piecewise linear (PWL) yield condition and constitutive laws.
    2. Nonlinear yield condition and piecewise linear (PWL) constitutive laws.
    3. Nonlinear yield condition and constitutive laws.
- Case (b): **NQM** interaction with:
    1. Piecewise linear (PWL) yield condition and constitutive laws.
    2. Nonlinear yield condition and piecewise linear (PWL) constitutive laws.
    3. Nonlinear yield condition and constitutive laws.

For this purpose, one steel frame is examined for the aforementioned cases and the corresponding results are presented below. Herein the generalized Gendy-Saleeb yield criterion is adopted for both cases of interaction

[17]. It is noted that all analysis results of this method are presented following the engineering sign convention of structural analysis. Furthermore, for the case of the PWL yield condition and constitutive laws the cone identification approach is adopted [14,15].

The example concerns a three-storey, two-bay steel frame shown in Fig. 4. The frame is discretized into 30 elements, 21 nodes and 54 degrees of freedom. The steel grade is S235 with E=2×10^{8}kN/m^2. The material properties are as follows: sections with $A=112.5\times10^{-4}\ m^2$, $I=18260\times10^{-8}\ m^4$, $s_{1y}=2643.75kN$, $v_y=505.41kN$, $s_{2y}=s_{3y}=325kNm$ are employed for all columns, sections with $A=28.48\times10^{-4}\ m^2$, $I=1943\times10^{-8}\ m^4$, $s_{1y}=669.28kN$, $v_y=189.89kN$, $s_{2y}=s_{3y}=51.84kNm$ for all beams. The corresponding multi-segment hardening behavior is shown in Fig. 5a and depends on the parameters of every section. More specifically, for columns $h_1=6500kNm$ $z_1=0.005$ $\lambda_1=1.1$, $h_2=3250kNm$ $z_2=0.015$ $\lambda_2=1.20$, $h_3=-5200kNm$ $z_3=0.04$ $\lambda_3=0.8$, $h_4=10^{-6}kNm$ $z_4=0.05$ $\lambda_4=0.8$, while for beam cross sections $h_1=518.4kNm$ $z_1=0.005$ $\lambda_1=1.05$, $h_2=259.2kNm$ $z_2=0.015$ $\lambda_2=1.10$, $h_3=-777.6kNm$ $z_3=0.035$ $\lambda_3=0.80$, $h_4=10^{-6}kNm$ $z_4=0.04$ $\lambda_4=0.80$. The nonlinear structural behavior for column and beam cross sections is depicted in Fig. 5b using 4th degree polynomial lines, based on data presented in Table 1. The values of z_4 constitute the upper bounds for column and beam cross sections respectively. The upper bound vector of all displacements is $u_u=1$ and the lower bound vector $u_l=-1$ (for case b$_1$ the corresponding bounds are set as 2 and -2). An updating rule of $\rho=10\rho$ after each NLP solution until an appropriate convergence tolerance is reached $(w^T z \leq 10^{-5})$.

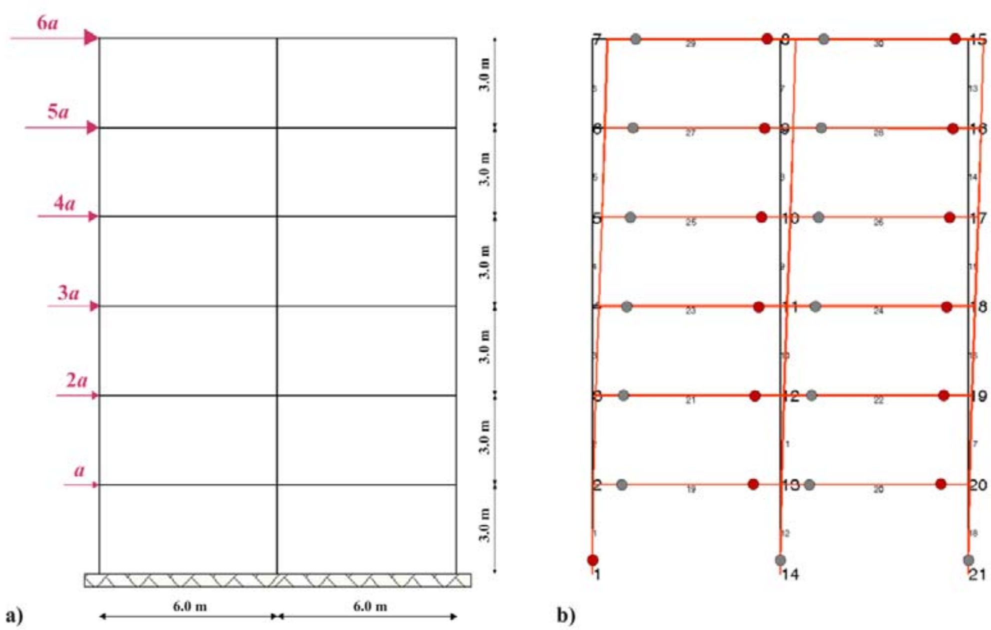

Figure 4: a) Three-storey, two-bay steel frame and b) plastic hinge formation for all analysis cases.

x	0.00	0.001	0.002	0.005	0.007	0.01	0.015	0.020	0.025	0.028	0.030	0.035	0.038	0.040	0.045	0.050
f(x)	1.00	1.05	1.10	1.15	1.18	1.20	1.18	1.10	1.05	1.00	1.10	0.95	0.90	0.85	0.80	0.80
Polynomial line for column cross sections								$f(x)=p_1\cdot x^4+p_2\cdot x^3+p_3\cdot x^2+p_4\cdot x+p_5$ $p_1=-2.17\cdot10^5, p_2=4.19\cdot10^4, p_3=-2411, p_4=38.55, p_5=1.014$								
x	0.00	0.0005	0.001	0.005	0.008	0.015	0.017	0.019	0.020	0.021	0.023	0.025	0.028	0.030	0.035	0.050
f(x)	1.00	1.01	1.03	1.05	1.08	1.10	1.09	1.07	1.05	1.02	1.00	0.95	0.90	0.87	0.83	0.80
Polynomial line for beam cross sections								$f(x)=p_1\cdot x^4+p_2\cdot x^3+p_3\cdot x^2+p_4\cdot x+p_5$ $p_1=2.38\cdot10^5,\ p_2=-4310,\ p_3=-827,\ p_4=18.24,\ p_5=0.9998$								

Table 1. Polynomial lines of structural behavior.

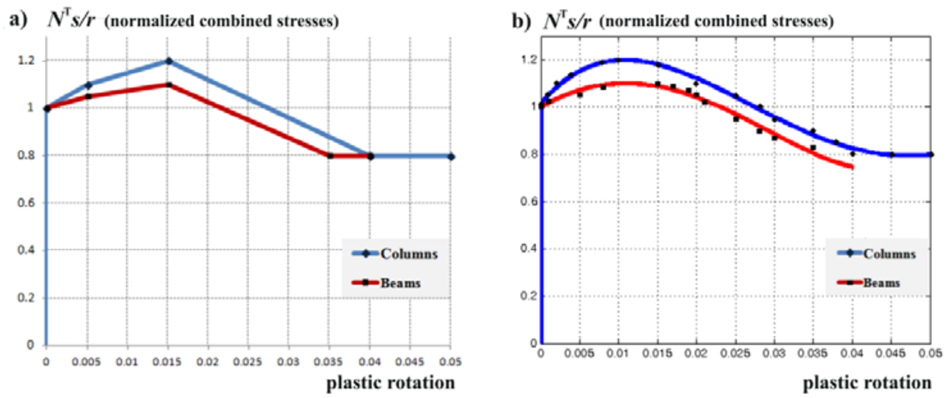

Figure 5: a) Multi-linear and b) nonlinear hardening/softening structural behavior.

All analysis results are presented in Table 2. Load factors of nonlinear approach for both interaction considerations are slightly greater than that of PWL approach, as expected. Moreover, the effect of shear force is evident in the reduction of the load carrying capacity. The plastic hinge pattern is identical for all analysis cases (Fig. 4b), but different stress states correspond to plastic hinges for each case. The corresponding interaction diagrams are shown in Figs. 6 and 7. Cross sections are stressed mainly due to bending moment, with some beam sections lying on their softening branch. The effect of axial force is observed mainly in beam cross sections (Fig. 6 and *n-m* diagrams of Fig. 7), while the effect of shear force is obvious and more intense than that of axial force in both beam and column cross sections (*v-m* diagrams of Fig. 7).

Cases	Case (a_1)	Case (a_2)	Case (a_3)	Case (b_1)	Case (b_2)	Case (b_3)
number of variables n_{var}	**205**					
number of equality constraints n_{eq}	**144**					
number of inequality constraints n_{inq}	**60**					
maximum load factor a (kN)	**8.24**	**8.28**	**8.35**	**7.90**	**8.25**	**8.31**
number of plastic hinges	**27**	**27**	**27**	**27**	**27**	**27**
total computational time (s)	**18.82**	**483.61**	**542.47**	**825.10**	**536.51**	**715.52**
number of iterations	**59**	**80**	**84**	**102**	**83**	**95**
complementarity condition $w^T z$	**7.35E-12**	**1.84E-10**	**4.56E-11**	**1.82E-10**	**9.40E-13**	**1.50E-12**
initial values of ρ	**10**	**10**	**10**	**10^5**	**100**	**10**

Table 2: Analysis results for all cases.

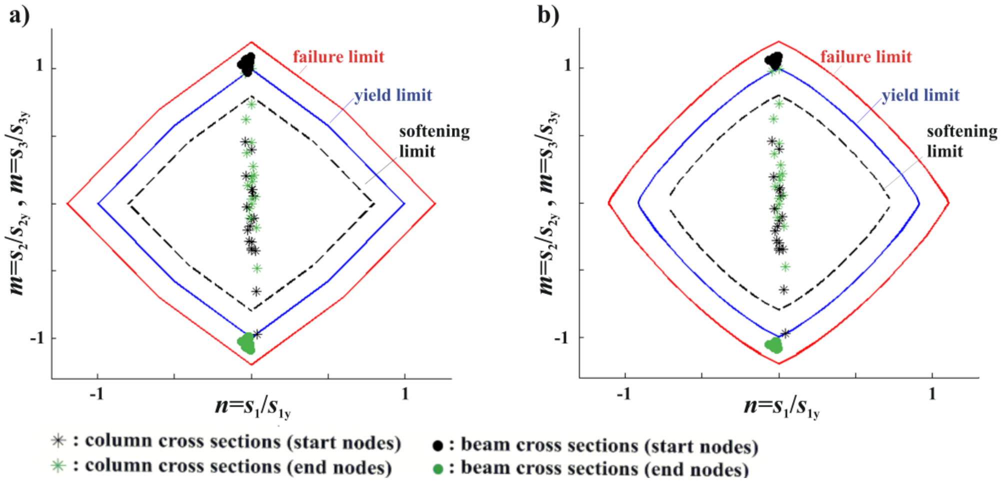

Figure 6: NM interaction diagrams for a) case (a_1) and b) case (a_3).

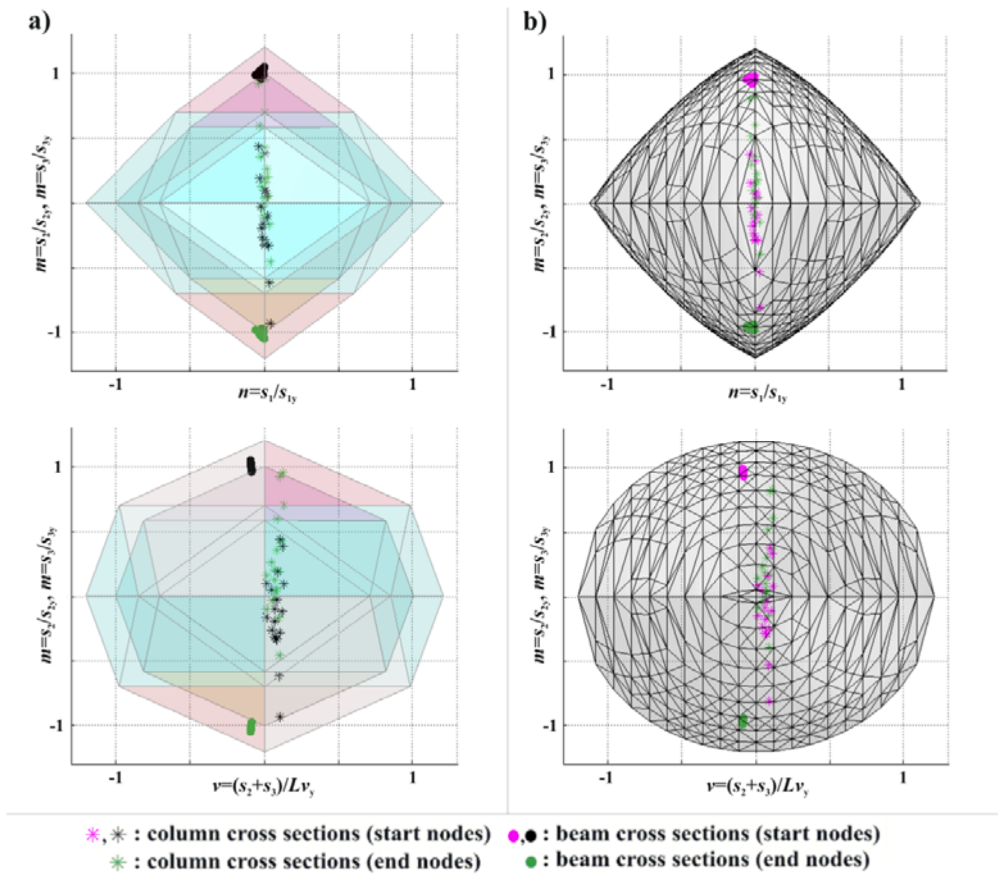

Figure 7: NQM interaction diagrams for a) case (b_1) and b) case (b_3).

The computational performance for NM and NQM interaction is presented in Fig. 8 and 9 respectively, omitting initial iterations for demonstration purposes. It is observed that for NM interaction the algorithm for all formulations seems to follow almost the same path. However, the nonlinear formulations (cases a_2 and a_3) need more iterations and consequently significantly more computational time compared to the PWL approach (case a_1). For NQM interaction, the optimization procedure of the local linearization procedure (cases b_2 and b_3) requires fewer iterations compared to the PWL one (83 and 95 iterations versus 102) and less computational time (536.51s, 715.52s versus 825.10s). This is due to the fact that cone identification procedure for 3D interaction consumes more computational time compared to local linearization of the yield surface. The sharp peaks that are presented for case (b_1) are due to the penalized term of the complementarity condition in the objective function. The generated vector of variables ***s*** and **z** determine a dot product $\boldsymbol{w}^T\mathbf{z}$ that deviates slightly from zero, but this is magnified by the penalty parameter ρ affecting noticeably the value of the objective function.

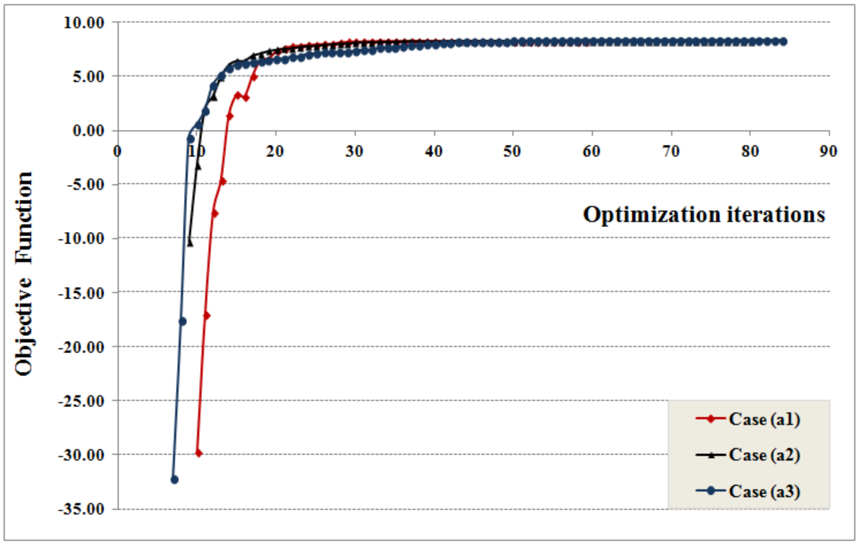

Figure 8: Evolution of the optimization procedure for NM interaction.

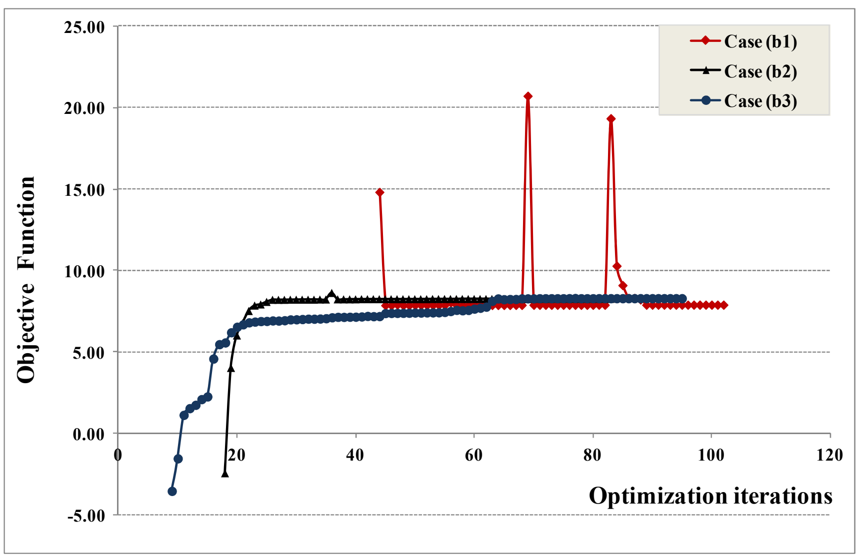

Figure 9: Evolution of the optimization procedure for NQM interaction.

4 CONCLUSIONS

Limit load and deformation analysis is treated herein in the framework of mathematical programming. The ultimate state of a structure and its maximum load carrying capacity is determined by solving an optimization problem that aims at maximizing the load factor a subjected to constraints that enforce equilibrium, compatibility, yielding and complementarity conditions. Due to the disjunctive nature of the latter, the problem lacks in convexity and smoothness. Using a penalty function method, it is reformulated into a NLP problem depending on initial values and lower and upper bounds of variables. The optimization process follows a gradient-based mathematical pace that tends to increase the load factor a satisfying the imposed constraints at each optimization iteration. Successive trials determine the path to the solution that generally differs from the actual path of a step-by-step method of finite element analysis, both though succeeding in finding the same solution.

In this work, limit load and deformation analysis of plane frames under holonomic consideration is dealt as a nonlinear programming problem incorporating appropriately nonlinear interaction and constitutive laws. The proposed method is based on the local linearization of the yield surface, while constitutive laws are embedded retaining their nonlinearity. The whole formulation preserves the linear formulation of the yield condition and reduces its size to a minimum (compared to the standard formulation) of both yield and complementarity conditions. At each optimization iteration and for every cross section the targeted or activated yield hyperplane is determined by locally linearizing the yield surface. Nonlinear structural behavior is also incorporated without affecting the linearity or the size of yield condition. From the example presented, more accurate solutions compared to PWL method are provided avoiding the cumbersome procedure of the a priori linearization of the yield surface and constitutive laws.

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