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NUMERICAL INTEGRATION OF A NEW SET OF EQUATIONS OF MOTION FOR MECHANICAL SYSTEMS WITH SCLERONOMIC CONSTRAINTS

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Abstract. *Some new theoretical and numerical results are presented on the dynamic response of a class of mechanical systems with equality motion constraints. At the beginning, the equations of motion of the corresponding unconstrained system are presented, first in strong and then in a weak form. Next, the formulation is extended to systems with holonomic and/or nonholonomic constraints. The formulation is based on a new set of equations of motion, represented by a system of second order ordinary differential equations (ODEs) in both the coordinates and the Lagrange multipliers associated to the motion constraints. Moreover, the position, velocity and momentum type quantities are assumed to be independent, forming a three field set of equations. The weak formulation developed was first used to cast the equations of motion as a set of first order ODEs in the coordinates and the corresponding momenta. Then, the same formulation was also employed as a basis for producing a suitable time integration scheme for the systems examined. The validity and efficiency of this scheme was tested and illustrated by applying it to a number of characteristic example systems.*

1 INTRODUCTION

New results on the dynamics of constrained mechanical systems are of great interest to the engineering community since they are useful in many areas, including mechanisms, robotics, machinery, biomechanics, automotive and aerospace structures (e.g., [1-4]). Typically, the equations of motion for this class of systems are derived and cast in the form of a set of differential-algebraic equations (DAEs) of high index [3, 5, 6]. However, both the theoretical and the numerical treatment of DAEs is a delicate and difficult task [7]. For this reason, many attempts have been performed in the past in an effort to cure the problems related to a DAE modeling [8].

One way to avoid difficulties associated with the DAE formulation is to impose an appropriate scaling on the equations of motion and the generalized coordinates [9]. Another class of methods is based on differentiation of the constraint equations. In this way the index of the original DAEs is reduced to 2 or 1 [3, 5]. However, these methods need a stabilization since they lead to constraint violation in the lower kinematical levels [10]. Finally, another general category of methods is based on elimination of the Lagrange multipliers in order to convert the original DAEs to second order ODEs, by selecting a suitable set of independent generalized coordinates or velocities [6, 11]. This process is inconvenient to apply in practical problems since it is only locally valid. Moreover, other methods involving elimination of dependent velocities or the Lagrange multipliers do not possess an inherent mechanism to avoid constraint drift in the lower kinematical levels [12, 13].

The present formulation was based on a new set of equations of motion, represented by a coupled system of second order ODEs in both the coordinates of the system and the Lagrange multipliers associated to the motion constraints [14]. Here, these equations are first put in a convenient weak form. Moreover, the position, velocity and momentum type quantities were assumed to be independent, forming a three field set of equations [15-17]. In particular, the weak velocities and the strong time derivatives of all the coordinates involved in the formulation were related through a new set of Lagrange multipliers, which were shown to represent momentum type variables. The weak formulation developed was first used to cast the equations of motion as a set of first order ODEs in the coordinates and the corresponding momenta, resembling the structure of the classical Hamilton's canonical equations [18]. Alternatively, the same formulation can also be employed as a basis for producing suitable time integration schemes for the class of systems examined. One such scheme was developed for the purposes of this study. The validity and efficiency of this scheme was tested and illustrated by applying it to a number of characteristic example mechanical systems. Among other things, the results obtained verify that the scheme developed passes successfully all the tests related to a special set of challenging benchmark problems, chosen by the multibody dynamics community [19].

The organization of this paper is as follows. First, the strong form of the equations of motion governing the dynamics of an unconstrained discrete mechanical system is presented briefly in the following section. Then, the corresponding weak form of these equations is derived in the third section. Next, the weak form of the equations of motion for the class of constrained multibody systems examined is derived and presented in the fourth section. Based on this form, the equations of motion are then derived as a set of first order ODEs in the generalized coordinates and the corresponding momenta. Finally, a temporal discretization scheme was also developed and numerical results were obtained for several characteristic examples. These results are presented in the fifth section. The work is completed by a summary of the main findings.

2 STRONG FORM OF NEWTON'S LAW ON A MANIFOLD

This work refers to dynamics of a class of mechanical systems whose position can be determined by specifying a finite number of generalized coordinates, say $q^1,\dots,q^n$, at any time instance t [18]. The motion of such a system can be represented by the motion of a fictitious point, say p, along a curve $\gamma=\gamma(t)$ in an n-dimensional manifold M, the configuration space of the system [4, 20]. Moreover, the tangent vector $\underline{v}=d\gamma/dt$ to this curve belongs to an n-dimensional vector space, the tangent space of the manifold at p, denoted by T_pM [21].

By construction, for any point p of M, a coordinate map φ can be defined, acting from a neighborhood of p on M to the Euclidean space $\mathbb{R}^n$. Then, any point p on M is mapped through this map to a point

$$q=\varphi(p) \tag{1}$$

of $\mathbb{R}^n$, with $q=(q^1,\dots,q^n)$ representing the coordinates of p [22]. Then, by adopting the usual summation convention on repeated indices [20], if $c(t)$ represents a curve on $\mathbb{R}^n$, its tangent vector at point $q(t)$ of $c(t)$ is

$$\underline{\dot{q}}(t)=\dot{q}^i(t)\underline{G}_i\,, \tag{2}$$

where $\mathfrak{B}_G=\{\underline{G}_1 \quad \dots \quad \underline{G}_n\}$ is a basis of $\mathbb{R}^n$. As usual, the base vectors $\underline{G}_i$ remain constant throughout the flat space $\mathbb{R}^n$. Moreover, one can always define a coordinate basis $\mathfrak{B}_g=\{\underline{g}_1 \quad \dots \quad \underline{g}_n\}$ for space T_pM by

$$\underline{g}_i=\varphi_*^{-1}\underline{G}_i, \qquad i=1,\dots,n\,, \tag{3}$$

where φ_*^{-1} represents the inverse of the tangent mapping φ_*.

In Analytical Dynamics, it is frequently useful to employ a basis of T_pM, say $\mathfrak{B}_e=\{\underline{e}_1 \quad \dots \quad \underline{e}_n\}$, which is different than $\mathfrak{B}_g$ [23]. This new basis is obtained by the basis $\mathfrak{B}_g$ through a convenient linear transformation

$$\underline{e}_L=A_L^i\underline{g}_i, \qquad L,i=1,\dots,n \tag{4}$$

Then, any element $\underline{u}$ of the vector space T_pM can be expressed in the equivalent forms

$$\underline{u}=u^L\underline{e}_L=u^i\underline{g}_i\,, \tag{5}$$

with

$$u^i=A_L^iu^L \quad \text{and} \quad u^L=B_i^Lu^i \tag{6}$$

and

$$B_i^LA_M^i=\delta_M^L \quad \text{and} \quad B_i^LA_L^j=\delta_i^j\,, \tag{7}$$

where δ_j^i is a Kronecker's delta [22]. In addition, any such basis is characterized by its structure constants c_{IJ}^K, defined through the Lie bracket in the form

$$[\underline{e}_I,\underline{e}_J]=c_{IJ}^K\underline{e}_K\,. \tag{8}$$

In such cases, one can write

$$u^L=\dot{\theta}^L \tag{9}$$

but the quantities θ^L may not be true coordinates. Then, the last relation, in conjunction with Eq. (6), leads naturally to the useful concepts of quasi-coordinates and quasi-velocities [23]. Although the quasi-coordinates do not appear explicitly in the expressions used, their derivatives appear and are defined clearly in terms of the corresponding true coordinates. For instance, if f is a function on manifold M, then

$$\frac{\partial f}{\partial\theta^L}=\frac{\partial f}{\partial q^i}\frac{\partial q^i}{\partial\theta^L}=A_L^i\frac{\partial f}{\partial q^i}\,. \tag{10}$$

Based on the above, the derivative of a function f on manifold M along a vector field $\underline{u}$ can be determined by

$$\underline{u}(f) = f_{,i}u^i = f_{,L}u^L , \tag{11}$$

where the terms $f_{,i}$ and $f_{,L}$ represent the partial derivative of function f with respect to q^i and θ^L, respectively. This means that one can simplify the notation, by dropping the upper case indices in the sequel, provided that it is clear what is the basis employed for T_pM. For instance, the Lie derivative of a vector field $\underline{u}$ with respect to another vector field $\underline{w}$ on the configuration manifold M is given by

$$\mathfrak{L}_{\underline{w}}\underline{u} = [\underline{w},\underline{u}] = (w^j u^i_{,j} - u^j w^i_{,j} + c^i_{jk} w^j u^k)\underline{e}_i , \tag{12}$$

where $u^i_{,j}$ represents the derivative of u^i along the base vector $\underline{e}_j$ [22].

Likewise, the rate of change of a vector field $\underline{u}(t)$ on M along a curve with tangent vector $\underline{v}$ is determined by the covariant differential of $\underline{u}(t)$ along $\underline{v}$, having the form

$$\nabla_{\underline{v}}\underline{u}(t) = (\dot{u}^k + \Lambda^k_{ij} v^i u^j)\underline{e}_k , \tag{13}$$

where ∇ is the affine connection of the manifold. A more common notation in engineering literature [20] is

$$\nabla_{\underline{v}}\underline{u}(t) = \tfrac{Du^k}{Dt}\,\underline{e}_k = u^k\,|_j\, v^j \underline{e}_k ,$$

with components

$$u^k\,|_j \equiv u^k_{,j} + \Lambda^k_{ji} u^i . \tag{14}$$

The components Λ^k_{ij} of the connection ∇ in the basis of T_pM, known as affinities, are defined by

$$\nabla_{\underline{e}_i}\underline{e}_j = \Lambda^k_{ij}\underline{e}_k . \tag{15}$$

Finally, one can define the dual space to T_pM, denoted by T^*_pM, with elements known as covectors. In dynamics, a correspondence between a covector $\underset{\sim}{u}^*$ and a vector $\underline{u}$ is established through the dual product

$$\underset{\sim}{u}^*(\underline{w}) \equiv \langle \underline{u}, \underline{w} \rangle, \quad \forall \underline{w} \in T_pM , \tag{16}$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product of vector space T_pM [22]. In this way, to each basis $\{\underline{e}_i\}$ (with $i = 1,\ldots,n$) of T_pM, a dual basis $\{\underset{\sim}{e}^i\}$ can be established for T^*_pM by employing the condition

$$\underset{\sim}{e}^i(\underline{e}_j) = \delta^i_j . \tag{17}$$

Then, the covariant differential of a covector field $\underset{\sim}{u}^*(t)$ on M along a vector $\underline{v}$ of T_pM is evaluated by

$$\nabla_{\underline{v}}\underset{\sim}{u}^*(t) = (\dot{u}_k - \Lambda^j_{ik} v^i u_j)\underset{\sim}{e}^k . \tag{18}$$

Determination of the true path of motion (or natural trajectory) on a manifold is based on application of Newton's second law in the form

$$\nabla_{\underline{v}}\underset{\sim}{p}^*_M = \underset{\sim}{f}^*_M , \tag{19}$$

where $\underline{v}$ is the tangent vector of the natural trajectory $\gamma(t)$, while $\underset{\sim}{f}^*_M = f_i \underset{\sim}{e}^i$ represents the applied force [4, 24]. Moreover, the generalized momentum is defined as the covector corresponding to the velocity vector, i.e.,

$$\underset{\sim}{p}^*_M \equiv \underset{\sim}{v}^* .$$

Then, if $\underline{v} = v^i \underline{e}_i$ and $\underset{\sim}{p}^*_M = p_i \underset{\sim}{e}^i$, application of Eq. (16) leads to

$$p_i = g_{ij} v^j , \tag{20}$$

where the quantities

$$g_{ij} = \langle \underline{e}_i, \underline{e}_j \rangle \tag{21}$$

represent the components of the metric tensor at point p. As usual, these quantities are selected to coincide with the elements of the mass matrix G of the system, defined through the kinetic energy [20].

3 WEAK FORM OF NEWTON'S LAW ON A MANIFOLD

Through the definition of a class of special covectors (called Newton covectors, see [14]) by

$$\underset{\sim}{h}^*_M \equiv \nabla_{\underline{v}}\underset{\sim}{p}^*_M - \underset{\sim}{f}^*_M , \tag{22}$$

the equations of motion (19) at any point on a configuration manifold M can be put in the form

$$\underset{\sim}{h}_M^* = \underset{\sim}{0}\,. \tag{23}$$

Therefore, when there exist no motion constraints, it should be true that

$$\underset{\sim}{h}_M^*(\underline{w}) = 0\,, \tag{24}$$

for any element $\underline{w}$ of the tangent space T_pM . This, in turn, implies that

$$\int_{t_1}^{t_2} \underset{\sim}{h}_M^*(\underline{w})dt = 0, \quad \forall \underline{w} \in T_p M \tag{25}$$

along a natural trajectory on the manifold and within any time interval $[t_1, t_2]$.

Manipulation of the last integral requires application of integration by parts of the covariant derivative appearing in Eq. (22). This is achieved by employing the relation

$$\nabla_{\underline{v}}(\underset{\sim}{p}_M^*(\underline{w})) = (\nabla_{\underline{v}}\underset{\sim}{p}_M^*)(\underline{w}) + \underset{\sim}{p}_M^*(\nabla_{\underline{v}}\underline{w})\,, \tag{26}$$

which can be interpreted as a Leibniz rule on differentiation [22]. Then, the following expression is obtained

$$\int_{t_1}^{t_2} [\nabla_{\underline{v}}(\underset{\sim}{p}_M^*(\underline{w})) - \underset{\sim}{p}_M^*(\nabla_{\underline{v}}\underline{w}) - \underset{\sim}{f}_M^*(\underline{w})]dt = 0\,. \tag{27}$$

Finally, after an integration by parts of the first term inside the integral, the last equation becomes

$$[\underset{\sim}{p}_M^*(\underline{w})]\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} [\underset{\sim}{p}_M^*(\nabla_{\underline{v}}\underline{w}) + \underset{\sim}{f}_M^*(\underline{w})]dt = 0\,. \tag{28}$$

This equation represents the so called weak form of the equations of motion [15, 16]. In essence, it constitutes an alternative way to determine the true history of the coordinates (i.e., position) and velocities of a mechanical system satisfying the law of motion, as expressed by Eq. (19) originally.

Further manipulation of the weak form of the equations of motion (28) involves differentiations along the vectors $\underline{v}$ and $\underline{w}$. This task requires the construction of two smooth vector fields on the configuration manifold M , based on these two vectors. The first of these vector fields can be constructed by considering the tangent vector $\underline{v}$ at each point of the natural trajectory $\gamma(t)$. The second vector field can then be created by introducing another vector $\underline{w}$ of the tangent space at each point of the same trajectory, which can be arbitrary. Therefore, based on the above, a variation of any scalar function f is defined as the derivative of f along vector $\underline{w}$, by

$$\delta f \equiv \underline{w}(f) = f_{,i}\, w^i\,. \tag{29}$$

Likewise, the differential of f is defined by

$$d f \equiv \underline{v}(f) = f_{,i}\, v^i\,. \tag{30}$$

For instance, the variation and the differential of q^i are

$$\delta q^i = q^i_{,j}\, w^j \quad \text{and} \quad dq^i = q^i_{,j}\, v^j\,. \tag{31}$$

4 WEAK FORM OF NEWTON'S LAW FOR SYSTEMS WITH EQUALITY MOTION CONSTRAINTS

Next, consider systems subject to an additional set of k scleronomic constraints, appearing in the form

$$\dot{\psi}(q, \underline{v}) \equiv A(q)\underline{v} = \underline{0}\,, \tag{32}$$

where $\underline{v}$ is a vector in T_pM and $A = [a_i^R]$ is a known $k \times n$ matrix. Considering each of the resulting scalar equations separately, these constraints can be viewed in the dual product form

$$\dot{\psi}^R(q, \underline{v}) \equiv (\underset{\sim}{a}^R(q))(\underline{v}) = 0\,,\ (R = 1, \ldots, k), \tag{33}$$

where the quantity $\underset{\sim}{a}^R(q)$ represents the R-th row of matrix $A(q)$ with elements $a^R_{\bullet i}(q)$ and is treated as a covector leaving in the cotangent space T_p^*M . In the special case where the constraint examined is holonomic, this condition can be integrated and written in the algebraic form

$$\phi^R(q) = 0\,. \tag{34}$$

Based on the above, the equations of motion of the class of systems examined can be put in the form

$$\underset{\sim}{h}_M^* = \underset{\sim}{h}_C^* \tag{35}$$

on the original manifold M [14], with

$$\underset{\sim}{h}_M^* = h_i \underset{\sim}{e}^i = [(g_{ij}v^j)^{\bullet} - \Lambda^m_{\ell i} g_{mj} v^j v^\ell - f_i]\underset{\sim}{e}^i \tag{36}$$

and

$$\underset{\sim}{h}_C^* = \sum_{R=1}^{k} [(\overline{m}_{RR}\dot{\lambda}^R)^{\bullet} + \overline{c}_{RR}\dot{\lambda}^R + \overline{k}_{RR}\lambda^R - \overline{f}_R] a_i^R \underset{\sim}{e}^i\,. \tag{37}$$

In the last relation, the convention on repeated indices does not apply to index R. Moreover, the coefficients

$$\overline{m}_{RR}=c_R^i g_{ij} c_R^j,\quad \overline{c}_{RR}=-c_R^i \frac{\partial f_i}{\partial v^j}(q,\underline{v},t)c_R^j,\quad \overline{k}_{RR}=-c_R^i f_{i,j}(q,\underline{v},t)c_R^j \tag{38}$$

represent an equivalent mass, damping and stiffness quantity, respectively, obtained through a projection along a special direction $\underline{c}_R$ on T_pM [14]. Specifically, the components of the n-vector $\underline{c}_R$ are selected from

$$c_R^i a_i^R = 1. \tag{39}$$

These components are also needed for determining the projected forcing term

$$\overline{f}_R = c_R^i f_i(q,\underline{v},t). \tag{40}$$

If generalized (true) coordinates are used, which means that $v^i=\dot{q}^i$, Eq. (35) represents a set of n second order coupled ODEs in the $n+k$ unknowns q^i and λ^R. The additional information needed for a complete mathematical formulation is obtained by incorporating the k equations of the constraints [14]. In particular, for each holonomic constraint, a second order ODE with form

$$(\overline{m}_{RR}\dot{\phi}^R)^{\cdot}+\overline{c}_{RR}\dot{\phi}^R+\overline{k}_{RR}\phi^R=0 \tag{41}$$

is obtained, which forces both $\dot{\phi}^R$ and ϕ^R to become zero eventually. Likewise, for each nonholonomic constraint

$$(\overline{m}_{RR}\dot{\psi}^R)^{\cdot}+\overline{c}_{RR}\dot{\psi}^R=0, \tag{42}$$

causing only $\dot{\psi}^R$ to become zero.

Taking into account the new set of equations of motion (35), Eq. (25) is first modified accordingly to

$$\int_{t_1}^{t_2}(\underset{\sim}{h}_M^*-\underset{\sim}{h}_C^*)(\underline{w})dt=0,\quad \forall \underline{w}\in T_pM. \tag{43}$$

Next, consider a holonomic constraint, as expressed by Eq. (41). It is easy to verify that the following is satisfied

$$\int_{t_1}^{t_2}[(\overline{m}_{RR}\dot{\phi}^R)^{\cdot}+\overline{c}_{RR}\dot{\phi}^R+\overline{k}_{RR}\phi^R]\delta\lambda^R dt=0,$$

for an arbitrary multiplier $\delta\lambda^R$. A similar expression is obtained for each nonholonomic constraint as well.

In a weak formulation, it is advantageous to consider the position, velocity and momentum variables as independent [17]. For this, a new velocity field $\underline{\upsilon}$ is introduced on manifold M, which should eventually be forced to become identical to the true velocity field through the components π_i of a covector, representing a set of Lagrange multipliers. A similar action can be taken for the velocity components $\dot{\lambda}^R$, by introducing another set of Lagrange multipliers, σ_R. Likewise, one can relate the strong time derivatives v^i and $\dot{\lambda}^R$ of the position type variables to weak velocities, υ^i and μ^R, through two new sets of Lagrange multipliers, denoted by $\delta\pi_i$ and $\delta\sigma_R$, respectively. To achieve these tasks, the weak form expressed by Eq. (43) should be augmented by the terms

$$\int_{t_1}^{t_2}[\pi_i(\delta\upsilon^i-\delta v^i)+\delta\pi_i(\upsilon^i-v^i)]dt \quad\text{and}\quad \int_{t_1}^{t_2}[\sigma_R(\delta\mu^R-\delta\dot{\lambda}^R)+\delta\sigma_R(\mu^R-\dot{\lambda}^R)]dt.$$

Finally, by adding up all these terms and performing appropriate mathematical operations yields eventually the following three field set of equations

$$\begin{aligned}
&(p_i-\textstyle\sum_{R=1}^k a_i^R\overline{m}_{RR}\mu^R-\pi_i)w^i\Big|_{t_1}^{t_2}+\textstyle\sum_{R=1}^k(\overline{m}_{RR}\dot{\phi}^R-\sigma_R)\delta\lambda^R\Big|_{t_1}^{t_2}\\
&+\int_{t_1}^{t_2}\delta\pi_i(\upsilon^i-v^i)dt+\textstyle\sum_{R=1}^k\int_{t_1}^{t_2}\delta\sigma_R(\mu^R-\dot{\lambda}^R)dt\\
&+\int_{t_1}^{t_2}[(-p_i+\textstyle\sum_{R=1}^k a_i^R\overline{m}_{RR}\mu^R+\pi_i)\delta\upsilon^i\\
&\qquad+\textstyle\sum_{R=1}^k(\sigma_R-\overline{m}_{RR}\dot{\phi}^R)\delta\mu^R]dt\\
&-\int_{t_1}^{t_2}\{(-p_\ell+\textstyle\sum_{R=1}^k a_\ell^R\overline{m}_{RR}\mu^R+\pi_\ell)(\tau_{ij}^\ell-\Lambda_{ij}^\ell+\sigma_{ij}^\ell)\upsilon^j\\
&\qquad+\textstyle\sum_{R=1}^k[(\overline{c}_{RR}\mu^R+\overline{k}_{RR}\lambda^R-\overline{f}_R)a_i^R-\overline{m}_{RR}\mu^R\frac{Da_i^R}{Dt}]\\
&\qquad+f_i-\frac{D\pi_i}{Dt}\}w^i dt\\
&+\textstyle\sum_{R=1}^k\int_{t_1}^{t_2}(\dot{\sigma}_R+\overline{c}_{RR}\dot{\phi}^R+\overline{k}_{RR}\phi^R)\delta\lambda^R dt=0,
\end{aligned} \tag{44}$$

where the variations w^i, $\delta\lambda^R$, $\delta\upsilon^i$, $\delta\mu^R$, $\delta\pi_i$ and $\delta\sigma_R$ are independent for all $i=1,\ldots,n$ and $R=1,\ldots,k$. Moreover,

$$\tfrac{D\pi_i}{Dt}=\pi_i\,|_j\,\upsilon^j=\dot{\pi}_i-\Lambda^{\ell}_{ji}\pi_\ell\upsilon^j \quad\text{and}\quad \tfrac{Da^R_i}{Dt}=a^R_i\,|_j\,\upsilon^j=\dot{a}^R_i-\Lambda^{\ell}_{ji}a^R_\ell\upsilon^j\,. \tag{45}$$

Equation (44) is the final weak form obtained for the class of constrained mechanical systems examined. This form is convenient for performing an appropriate numerical discretization, leading to improvements in existing numerical schemes based on advanced analytical tools (e.g., [25, 26]). It can also be used for obtaining an alternative form for the equations of motion (35), expressed as a set of first order ODEs in the coordinate and the corresponding conjugate momentum variables. Specifically, by first collecting terms multiplied with $\delta\pi_i$ yields

$$v^i=\upsilon^i\,, \tag{46}$$

which takes the more explicit form

$$\dot{q}^i=\upsilon^i \quad\text{or}\quad A^i_j\dot{\theta}^j=\upsilon^i \quad\Rightarrow\quad \dot{\theta}^i=B^i_j\upsilon^j\,,$$

when true or quasi-velocities are involved, respectively. Likewise, selecting terms of $\delta\sigma_R$ leads to

$$\dot{\lambda}^R=\mu^R\,. \tag{47}$$

Next, collecting terms multiplied by $\delta\upsilon^i$ yields

$$\pi_i=g_{ij}\upsilon^j-\sum\nolimits_{R=1}^{k}a^R_i\overline{m}_{RR}\mu^R\,, \tag{48}$$

while selecting terms of $\delta\mu^R$ leads to

$$\sigma_R=\overline{m}_{RR}\dot{\phi}^R\,. \tag{49}$$

The last two equations verify that the Lagrange multipliers π_i and σ_R appear as components of generalized momentum type quantities. Finally, collecting the terms multiplying w^i yields

$$\tfrac{D\pi_i}{Dt}=f_i+\sum\nolimits_{R=1}^{k}[(\overline{c}_{RR}\mu^R+\overline{k}_{RR}\lambda^R-\overline{f}_R)a^R_i-\overline{m}_{RR}\mu^R\tfrac{Da^R_i}{Dt}]\,, \tag{50}$$

while selecting terms of $\delta\lambda^R$ leads to

$$\dot{\sigma}_R=-\overline{c}_{RR}\dot{\phi}^R-\overline{k}_{RR}\phi^R\,. \tag{51}$$

For a nonholonomic constraint, Eqs. (49) and (51) become

$$\sigma_R=\overline{m}_{RR}\dot{\psi}^R \quad\text{and}\quad \dot{\sigma}_R=-\overline{c}_{RR}\dot{\psi}^R\,. \tag{52}$$

The set of Eqs. (48) and (49) can be viewed as a linear algebraic system in the weak velocities υ^i and μ^R. In principle, its solution furnishes these quantities as a function of the corresponding momentum quantities π_i and σ_R. Consequently, substitution of these results in Eqs. (46), (47), (50) and (51) yields a system of first order ODEs involving the coordinates q^i and λ^R together with the conjugate momenta π_i and σ_R as unknowns. The resulting set of equations is expressed in the cotangent bundle T^*M and possesses a similar structure but is in fact more general than the form of the classical Hamilton's canonical equations [18, 21]. Alternatively, direct substitution of Eqs. (48) and (49) into Eqs. (50) and (51) leads to a new system of first order ODEs involving the coordinates q^i and λ^R plus the velocities υ^i and μ^R as unknowns. Therefore, the resulting set of equations is now expressed in the tangent bundle TM, instead.

5 NUMERICAL RESULTS

The weak form derived in the previous section is convenient for developing efficient numerical discretization schemes for the new set of equations of motion employed, leading to improvements over existing schemes on constrained mechanical systems (e.g., [5, 25, 26]). For the purposes of the present work, this form was first put within the framework of an augmented Lagrangian formulation [27-29]. This leads to a full exploration of the major advantages of the theoretical method applied, in a quite natural manner [14]. More specifically, this method is appropriate for performing a geometrically exact discretization. This is especially useful when the configuration space of the system possesses group properties [30]. The success of this formulation was demonstrated by the accurate solution obtained for a number of challenging problems. Some characteristic results are presented next.

5.1 Example 1: Planar Simple Pendulum

The first mechanical system examined consists of a sphere of unit mass, possessing a mass moment of inertia matrix equal to the 3×3 identity matrix. Moreover, the center of this sphere is mounted at one end of a massless rigid bar of unit length. The system rotates about a fixed axis Oy, as shown in Fig. 1a, with an initial angular velocity of -100 rad/s. In Fig. 1b is depicted the history of the angular velocity ω_y over the first 30 seconds.

Next, results obtained by the present method are compared with those obtained by a state of the art code [31]. This code sets up the equations of motion as a system of DAEs and solves them by numerical integration. The results of the new method are labeled by LMD, while the results obtained by applying the stabilized index-1 technique of the code are labeled by SI1. Moreover, the numbers in the labels correspond to the error allowed in the calculations for each case. Quite similar results were also obtained by applying other DAE methods of the same code as well as by another state of the art code [32].

The results obtained by the DAE method deviate from the results of the new method, as shown in Fig. 1b. The discrepancy increases in a fast manner for solutions with smaller accuracy. Specifically, the magnitude of the angular velocity obtained by SI1 decreases with time, while the new approach captures the correct constant value. This deviation is reflected in other important quantities and is mostly due to the artificial damping involved in the calculations of the direct integration scheme employed by the code. For instance, a similar picture is obtained for the history of the kinetic energy of the system, as presented in Fig. 1c. Again, the new method captures the correct constant value of this quantity.

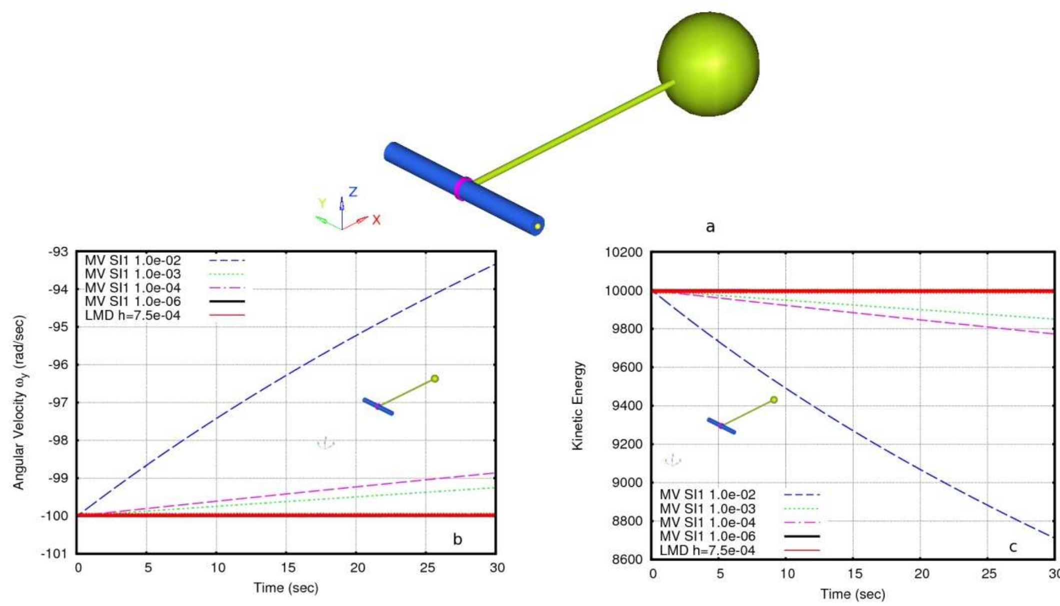
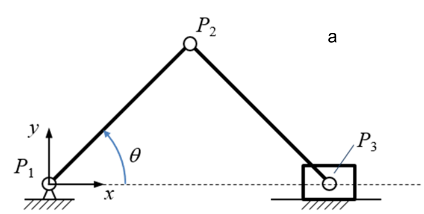


Fig. 1. (a) A rigid sphere-bar system in pure rotation. Comparison of time histories of the: (b) angular velocity, (c) kinetic energy of the system.

5.2 Example 2: Planar Slider-Crank Mechanism

A slider-crank system, shown in Fig. 2a, is considered next. It belongs to a special set of benchmark problems [19]. The two rods have equal length and uniformly distributed mass, while the slider has no mass and slides with no friction on the ground. Gravity acts along the negative y-axis. For $\theta = n\pi/2$, with $n = 0,1,\dots$, the system passes through a singular configuration.

First, in Fig. 2b are shown the time histories of the x and y coordinates of point P_2, while in Fig. 2c is depicted the mechanical energy of the system. Finally, in Figs. 2d and 2e are presented the corresponding histories of the constraint violations at the position and velocity levels, represented by the norm of the array of the constraints at each level.



Direct comparison of the results in Fig. 2 illustrates that the present method is accurate and passes successfully through the singular positions (Fig. 2b). It also presents better numerical performance. For instance, the mechanical energy computed by the present method remains constant virtually (Fig. 2c). In addition, the errors in both the displacement and velocity constraint violations are bounded and stay at the same level, throughout the time interval examined (Figs. 2d and 2e).

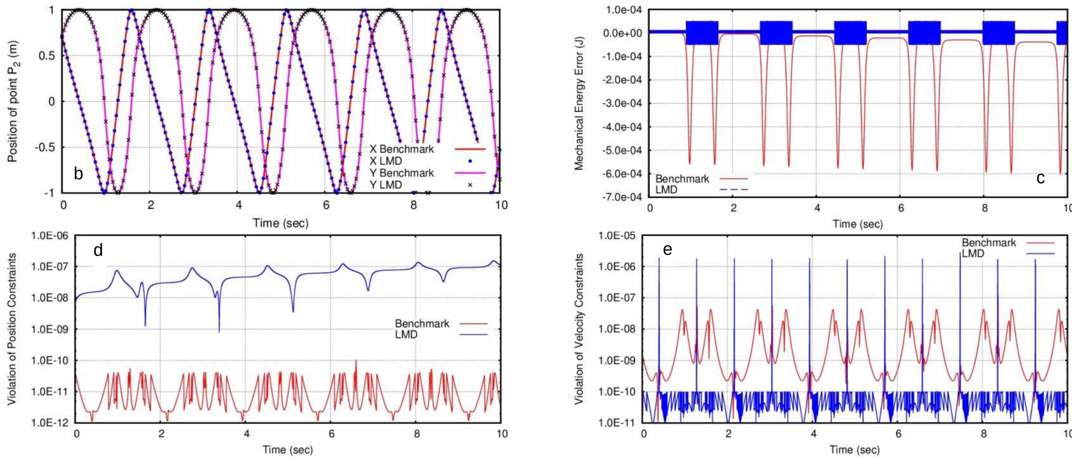


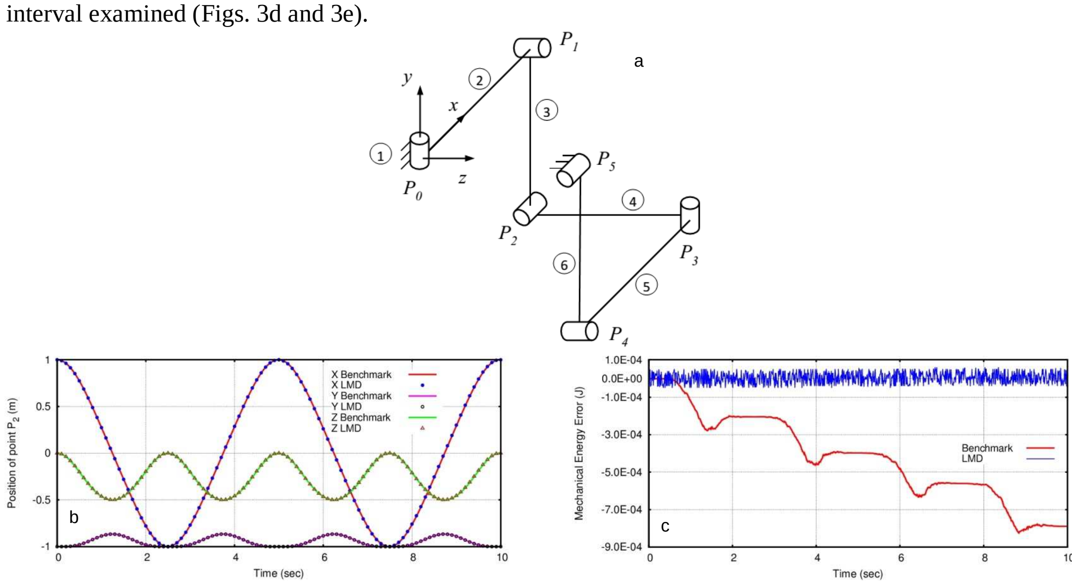
Fig. 2. (a) Mechanical model of a slider-crank mechanism, (b) x and y coordinates of point P_2, (c) mechanical energy, (d) violation of position and (e) violation of velocity constraints.

5.3 Example 3: Rectangular Bricard Mechanism

The next set of results refers to a six-bar rectangular Bricard mechanism, shown in Fig. 3a. All the rods are connected with revolute joints, have equal length and uniformly distributed mass [19]. Again, this system moves due to gravity acting along the negative y-axis. The mechanism examined represents a mechanical system which is redundantly constrained throughout its motion.

In Fig. 3b are shown the time histories of the x, y and z coordinates of point P_2, while in Fig. 3c is depicted the mechanical energy of the system. Finally, in Figs. 3d and 3e are presented the corresponding histories of the constraint violations at the position and velocity levels during the same time interval.

Once again, direct comparison of the results in Fig. 3 illustrates that the present method is accurate and passes successfully through the singular positions. It also presents an improved numerical performance. For instance, the mechanical energy computed by the present method remains virtually constant (Fig. 3c). In addition, the errors in both the displacement and velocity constraint violations are bounded and stay at the same level, throughout the time interval examined (Figs. 3d and 3e).



d

e

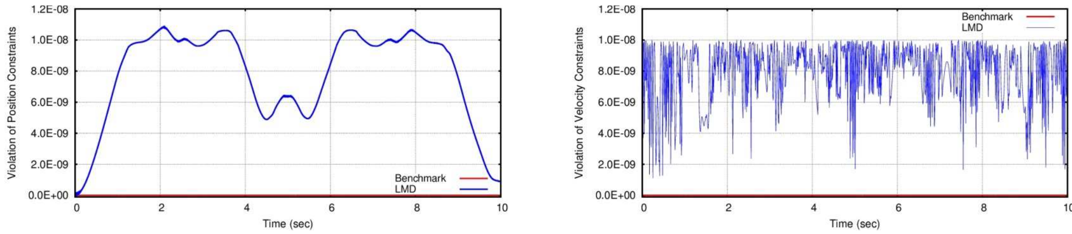


Fig. 3. (a) Mechanical model of a Bricard mechanism, (b) x, y and z coordinates of point P_2, (c) mechanical energy, (d) violation of position and (e) violation of velocity constraints.

5.4 Example 4: Andrews' Squeezer Mechanism

In the last mechanical example, shown in Fig. 4a, the planar, seven-link system known as Andrews' squeezer mechanism is examined [19, 33]. A constant driving torque acts on rod OF during its motion. The system moves without gravity effects and its technical parameters are such that a relatively small time step is required during the numerical integration of the equations of motion.

In Fig. 4b are shown the time histories of the x and y coordinates of point F, while in Fig. 4c is depicted the balance of the mechanical energy of the system. Finally, in Fig. 4d is presented the corresponding history of the constraint violations at the velocity level. Again, the results illustrate the accuracy and efficiency of the present method.

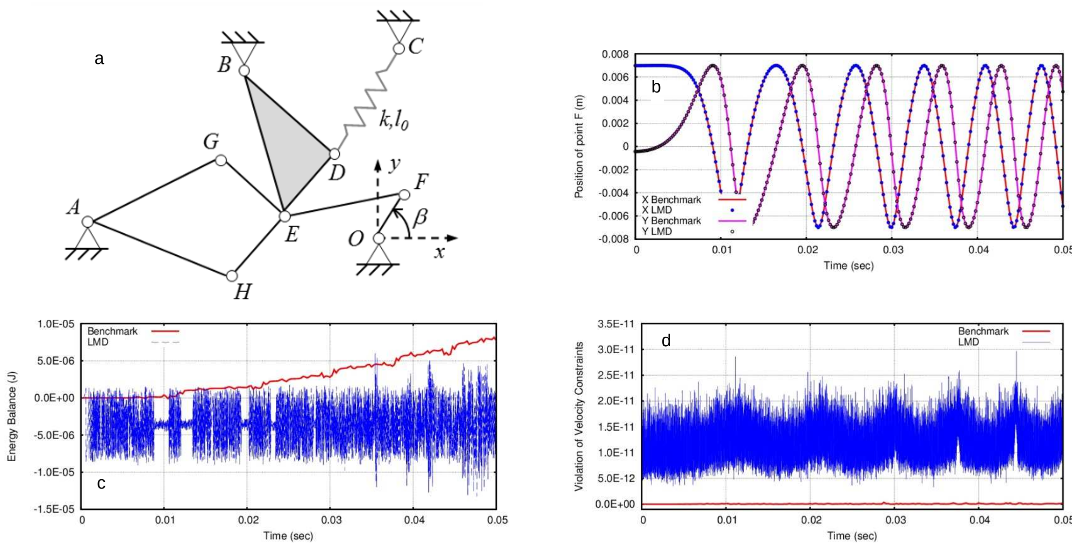


Fig. 4. (a) Mechanical model of Andrews' squeezer mechanism, (b) x and y coordinates of point F, (c) mechanical energy balance, (d) violation of velocity constraints.

6 SYNOPSIS

In the first part of this work, a weak form of the equations of motion for a class of mechanical systems was derived. Specifically, in addition to obeying Newton's law, the systems examined are subject to holonomic and/or nonholonomic scleronomic constraints. This formulation was based on a new set of equations of motion, represented by a coupled system of second order ODEs in both the generalized coordinates and the Lagrange multipliers associated to the motion constraints. Moreover, the position, velocity and momentum type quantities were assumed to be independent, forming a three field set of equations. The weak formulation developed was first used to cast the equations of motion to a set of first order ODEs in the coordinates and the corresponding momenta, resembling the structure of the classical Hamilton's canonical equations. Then, it was used as a basis for producing a suitable time integration scheme for the class of systems examined. The numerical accuracy and efficiency of this new scheme was demonstrated by presenting results for a selected set of mechanical examples.

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