SIMULATION OF PLANE STRAIN FIBER COMPOSITE PLATES IN BENDING THROUGH A BEM/ACA/HM FORMULATION

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Abstract. *In the present work large scale, plane strain elastic problems dealing with the bending of unidirectional fiber composite plates are solved by means of a Boundary Element method (BEM) accelerated via Adaptive Cross Approximation (ACA) and Hierarchical Matrices (HM) techniques. The composite plate is modeled as a large number of periodically or randomly distributed cylindrical elastic fibers embedded in a matrix medium. Each of the considered problems is treated through boundary discetizations with almost one million Degrees of Freedom (DoFs). The work aims to study microstructural effects due to the size of the fibers and the validity of the homogenization generalized self-consistent method proposed by Christensen (J. Mech. Phys. Solids, Vol. 38, pp. 379-404, 1990).*

1 INTRODUCTION

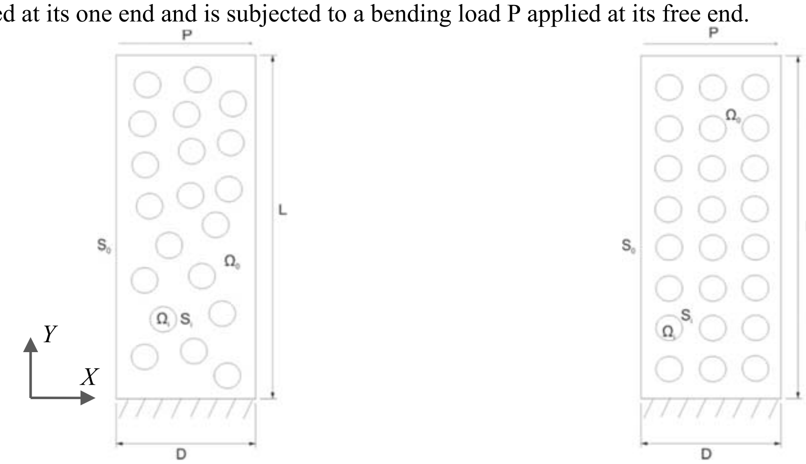
Composite materials are used in advanced mechanical structures with remarkable gain in strength and weight. However, due to their microstructural complexity, their computational modeling is still confined to small specimens corresponding to small numbers of inclusions. To overcome this difficulty, a plethora of theoretical and numerical homogenization techniques have been proposed so far in the literature predicting the effective properties of the composite. Most of them are based on the analysis of a unit cell consisting of an inclusion embedded in a matrix medium and obeying to specified boundary conditions. A widely used and very representative homogenization technique is the Generalized Self-Consistent Method (GSCM) proposed by Christensen [1] for fiber and particulate composite materials. The main goal of the present work is to find out the smallest size of a fiber composite the effective properties of which can be effectively predicted by the GSCM. To this end, the fiber composite material is simulated as a plate with a large number of periodically or randomly distributed unidirectional reinforcements subjected to a bending loading across the direction of the fibers.

The above described problem is numerically solved by means of a Boundary Element Method (BEM) accelerated via Adaptive Cross Approximation (ACA) and Hierarchical Matrices (HM) techniques (Bebendorf [2]). Conventional BEM formulations produces full populated and non-symmetric collocation matrices [A] increasing thus the computational cost and confining the method to the solution of relatively small problems. More precisely, the computation of all elements of [A] requires $O(N^2)$ algebraic operations, with N being the number of unknowns. Furthermore, the solution of the system of equations requires $O(N^3)$ operations if a direct solver is utilized and $O(K\times N^2)$ operations if an iterative solver is used, with K being the number of iterations. A solution to that problem is the use of a BEM enhanced by HM and ACA techniques that accelerate drastically the computation of matrix [A] and also reduce the memory requirements. That acceleration is possible due to the nature of the fundamental solutions, which are functions of the distance between the source and field points and thus only a small number of elements of the collocation matrix [A] are calculated, while the rest of them are approximated via the already evaluated elements~~.~~ According to ACA/BEM, the matrix [A] is organized into a hierarchical structure of blocks depended on the geometry of the problem. Applying a geometrical criterion the blocks are characterized either as non-admissible, where the ACA algorithm is inefficient and thus the conventional BEM is employed or admissible where ACA is effective and is used to calculate only a small number of their rows and columns. Each admissible block is represented in a low rank matrix format via the product of two matrices formed by the previously calculated rows and columns, respectively. This low rank format in conjunction with an iterative solver leads to significant reductions in memory requirements and CPU time due to the acceleration of the matrix vector multiplication. More details one can find in the works of Bebendorf and Grzhibovskis [3], Benedetti et al. [4], Benedetti et al. [5] and Zechner [6] for elastostatic problems and Benedetti and Alliabadi [7], Messner and Schanz [8] and Millazzo et al. [9] for elastodynamic ones.

2 PROBLEM DESCRIPTION AND SOLUTION METHOD

2.1 Problem description and conventional Boundary Element Method formulation

Consider a 2*D* rectangular fiber composite plate of length *L* and width *D*, ~~as~~ shown in figs. 1(a) and (b). The plate occupies a region Ω_0 of boundary S_0, ~~is~~ made of a matrix material with Young's modulus E_M and Poisson's ratio v_M and ~~is~~ reinforced ~~with~~ by N_f identical circular fibers of radius α with Young's modulus and Poisson's ratio E_F and v_F, respectively. The fibers are either randomly distributed (fig. 1(a)) or periodically arranged in a square pattern (fig. 1(b)), while their volume fraction is u_f. Each fiber occupies a region Ω_i of boundary S_i, where i= 1, N_f. The plate is fixed at its one end and is subjected to a bending load P applied at its free end.



(a) (b)

Figure 1. 2D rectangular plate with (a) randomly distributed and (b) periodically arranged fibers

The~~-~~solution of the above described 2D elastostatic problem can be obtained by solving a combined system of boundary integral equations written for the matrix and each of the N_f fibers. The boundary integral equations for the matrix and for the i^{th} fiber are written as:

$$\tilde{\mathbf{c}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+\int_S \tilde{\mathbf{t}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y})\,dS_{\mathrm{y}}=\int_S \tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{t}(\mathbf{y})\,dS_{\mathrm{y}} \tag{1}$$

$$\tilde{\mathbf{c}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+\int_{S_i} \tilde{\mathbf{t}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y})\,dS_{\mathrm{y}}=\int_{S_i} \tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{t}(\mathbf{y})\,dS_{\mathrm{y}} \tag{2}$$

where $S=S_0+\sum_{i=1}^{N_f}S_i$, **x** and **y** are points on the boundary, **u** and **t** are the displacement and traction vectors, $\tilde{\mathbf{c}}$ is a free term tensor depended on local geometry at point **x** (for a smooth boundary $\tilde{\mathbf{c}}=1/2\tilde{\mathbf{I}}$, with $\tilde{\mathbf{I}}$ being the unity tensor) and $\tilde{\mathbf{u}}^*(\mathbf{x},\mathbf{y})$ and $\tilde{\mathbf{t}}^*(\mathbf{x},\mathbf{y})$ are the 2D free space elastostatic fundamental solutions, written as [10].

$$u^*_{kj}(\mathbf{x},\mathbf{y})=\frac{(1+v)}{4\pi E(1-v)}\left((3-4v)\delta_{kj}\ln\left(\frac{1}{r}\right)+r_{,k}\,r_{,j}\right),\ \text{with}\ k,j=\ 1,\ 2 \tag{3}$$

$$t^*_{kj}(\mathbf{x},\mathbf{y})=\frac{1}{4\pi(1-v)r}\left[\left((1-2v)\delta_{kj}+2r_{,k}\,r_{,j}\right)\frac{\partial r}{\partial n}+(1-2v)\left(r_{,k}\,n_j-r_{,j}\,n_k\right)\right] \tag{4}$$

where r is the distance between the points $\mathbf{x}$ and $\mathbf{y}$, n_j the unit normal vector of the boundary at point $\mathbf{y}$, $r_{,j}$ denotes spatial derivatives of r and $\partial r / \partial n$ is the directional derivative with respect to the normal vector at $\mathbf{y}$.

According to a conventional BEM formulation, the boundary S is discretized into three-noded quadratic or two-noded linear isoparametric line boundary elements with a total number of L nodes. Collocating eqn (1) at all nodes L one obtains the following linear system of algebraic equations of the form

$$\begin{bmatrix} {}^M\tilde{\mathbf{H}}_0^0 & {}^M\tilde{\mathbf{H}}_1^0 & \cdots & {}^M\tilde{\mathbf{H}}_{N_f}^0 \\ {}^M\tilde{\mathbf{H}}_0^1 & {}^M\tilde{\mathbf{H}}_1^1 & \cdots & {}^M\tilde{\mathbf{H}}_{N_f}^1 \\ \vdots & \vdots & \vdots & \vdots \\ {}^M\tilde{\mathbf{H}}_0^{N_f} & {}^M\tilde{\mathbf{H}}_1^{N_f} & \cdots & {}^M\tilde{\mathbf{H}}_{N_f}^{N_f} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{u}^0 \\ \mathbf{u}^1 \\ \vdots \\ \mathbf{u}^{N_f} \end{Bmatrix} = \begin{bmatrix} {}^M\tilde{\mathbf{G}}_0^0 & {}^M\tilde{\mathbf{G}}_1^0 & \cdots & {}^M\tilde{\mathbf{G}}_{N_f}^0 \\ {}^M\tilde{\mathbf{G}}_0^1 & {}^M\tilde{\mathbf{G}}_1^1 & \cdots & {}^M\tilde{\mathbf{G}}_{N_f}^1 \\ \vdots & \vdots & \vdots & \vdots \\ {}^M\tilde{\mathbf{G}}_0^{N_f} & {}^M\tilde{\mathbf{G}}_1^{N_f} & \cdots & {}^M\tilde{\mathbf{G}}_{N_f}^{N_f} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{t}^0 \\ \mathbf{t}^1 \\ \vdots \\ \mathbf{t}^{N_f} \end{Bmatrix} \tag{5}$$

where ${}^M\tilde{\mathbf{G}}_\gamma^m$ and ${}^M\tilde{\mathbf{H}}_\gamma^m$ are matrices, formed by integrals containing the kernels (3) and (4), respectively, with the term 1/2 added at the diagonal elements of ${}^M\tilde{\mathbf{H}}_m^m$. The indices m and γ take values 0, 1, 2,…N_f, which correspond to the total number of nodes $L_0, L_1, ,.., L_{Nf}$ ($L = L_0 + \sum_{i=1}^{N_f} L_i$) that the boundaries S_0, $S_{1,\ ...,}$ S_{Nf} have been descritized into, respectively. The vectors $\mathbf{u}^{\mathrm{m}}$ and $\mathbf{t}^{\mathrm{m}}$ contain the nodal displacement and traction vectors of the nodes L_m, respectively.

Similarly, collocating the eqn (2), for all the nodes L_{i} of the i^{th} fiber and applying the continuity conditions (equal displacements and opposite tractions) at the interface between the matrix and the i^{th} fiber, the following system of algebraic equations can be obtained

$$\left[{}^F\tilde{\mathbf{H}}_i^i \right] \cdot \left\{ \mathbf{u}^i \right\} = -\left[{}^F\tilde{\mathbf{G}}_i^i \right] \cdot \left\{ \mathbf{t}^i \right\} \tag{6}$$

In systems of equations (5) and (6), all the nodal values of $\mathbf{u}^i$ and $\mathbf{t}^i$ with i=1, 2, ..,N_f are unknown, while the half nodal values of $\mathbf{u}^0$ and $\mathbf{t}^0$ are known by the boundary conditions on the boundary S_0 and the other half are unknown.

When point $\mathbf{x}$ does not coincide with $\mathbf{y}$, the integrals in eqs (1) and (2) are non-singular and can be easily computed numerically by Gauss quadrature. In case where $\mathbf{x}$ coincides with $\mathbf{y}$, the integrals in (1) and (2) become singular, with the integrals containing the kernel (3) being weakly singular of order $\mathrm{O}(\ln r)$ and the integrals containing the kernel (4) being strongly singular of order $\mathrm{O}(1/r)$. The singular integrals are evaluated with high accuracy by applying a direct integration method proposed by Guiggiani [11]. This direct evaluation makes use of a limiting process in the singular part of the kernels and then a semi-analytical integration is performed on a local co-ordinate system of the element, which has the origin at the singular point.

Combing eqs (5) and (6) and rearranging with respect to the boundary conditions valid at boundary S_0, one obtains a system of linear algebraic equations of the form

$$\tilde{\mathbf{A}} \cdot \mathbf{X} = \mathbf{B} \tag{7}$$

where the vectors $\mathbf{X}$ and $\mathbf{B}$ contain all the unknown and known nodal components of the boundary fields, respectively.

In the present work, the solution of the system (7) is obtained using the iterative solver GMRES. Actually, the matrix $\tilde{\mathbf{A}}$ is never formed explicitly, saving significant amount of memory which corresponds to the zero values appearing in $\tilde{\mathbf{A}}$ due to the fact that each fiber has a common interface only with the matrix medium and is not associated with the rest of the fibers. The GMRES multiplications are performed straightforward by considering eqs. (5) and (6). A block left diagonal preconditioner is used to accelerate the convergence of the solution. The dimensions of each block of the preconditioner are chosen to be equal to the number of degrees of freedom of a fiber, for a particular discretization. Each block is inverted by using the LU decomposition algorithm.

2.2 Hierarchical ACA accelerated BEM

In conventional BEM, the matrix $\tilde{\mathbf{A}}$ is generally a full populated matrix and thus, the memory demand is of $\mathrm{O}(N^2)$ which is prohibitive for solving realistic problems, where the degrees of freedom (DOFs) N are of the order of hundreds of thousands. In the present work, in order to overcome the conventional BEM memory limitations and solve the above described problem for a large number of fibers, a hierarchical ACA accelerated BEM is used. Furthermore, a significant reduction of the solution time is also accomplished.

According to the proposed method, the matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$, appearing in eqs (5) and (6), are represented hierarchically using a block tree structure. Under simple geometric considerations the blocks, which correspond to large distances r in kernels (3) and (4), are characterized as far field blocks (or admissible) and they are compressed by means of low rank matrices found by ACA, with respect to a prescribed accuracy ε. The rest blocks of the tree, which are dominated by the singular behavior of the kernels (3) and (4), are characterized as near field blocks (or non-admissible) and are calculated as in conventional BEM.

Let's consider an admissible block sub-matrix $\tilde{\mathbf{M}}$ of matrices $\tilde{\mathbf{H}}$ or $\tilde{\mathbf{G}}$, with dimensions $N \times L$ and a full rank $R = \min\{N, L\}$. The block $\tilde{\mathbf{M}}$ can be represented as:

$$\tilde{\mathbf{M}} = \tilde{\mathbf{M}}^{(K)} + \tilde{\mathbf{R}}^{(K)} \tag{8}$$

where $\tilde{\mathbf{M}}^{(K)}$ is a K-rank approximation of $\tilde{\mathbf{M}}$, with k being less equal than R and $\mathbf{R}^{(K)}$ is the residual of the approximation. $\tilde{\mathbf{M}}^{(K)}$ can be written as:

$$\tilde{\mathbf{M}}^{(K)} = \sum_{i=1}^{K} \mathbf{a}^i (\mathbf{b}^i)^T = \tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}^T \tag{9}$$

where $\mathbf{a}^i$ and $\mathbf{b}^i$ are vectors, of dimensions N and L, respectively, found such that the following relation holds:

$$\left\|\mathbf{R}^{(K)}\right\|_F \leq \varepsilon \left\|\tilde{\mathbf{M}}\right\|_F \tag{10}$$

where $\|\cdot\|_F$ denotes the Frobenius norm [2]. Matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$, of dimensions $N \times K$ and $L \times K$, respectively are formed by the vectors $\mathbf{a}^i$ and $\mathbf{b}^i$ as follows:

$$\begin{aligned} \tilde{\mathbf{A}}_{N\times K} &= \left\{\mathbf{a}^1 \quad \mathbf{a}^2 \quad \ldots \quad \mathbf{a}^3\right\} \\ \tilde{\mathbf{B}}_{L\times K} &= \left\{\mathbf{b}^1 \quad \mathbf{b}^2 \quad \ldots \quad \mathbf{b}^3\right\} \end{aligned} \tag{11}$$

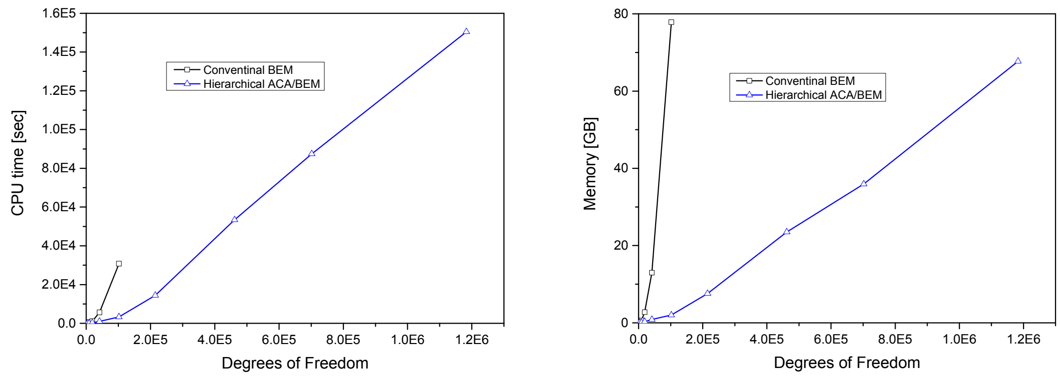
The memory requirements and matrix multiplication CPU cost of a low rank block are $O\left(K\left(N+L\right)\right)$, while for the corresponding full rank representation are $O(N \times L)$. It is obvious that the low rank approximation is efficient when the condition $K(N+L) << N \cdot L$ is true. The best low rank approximation for a given accuracy ε, can be found by means of the Singular Value Decomposition(SVD) [2].

Although the best low rank approximation (the minimal rank K) can be achieved via SVD, its cubic CPU cost with respect to the rank of the matrix. is prohibitive for real world applications. Thus, instead of the SVD, the Adaptive Cross Approximation (ACA is used. The main idea of ACA is to construct a representation of $\tilde{\mathbf{M}}^{(K)}$ (eqn 9) by suitably choosing a small subset of the rows and columns of a matrix $\tilde{\mathbf{M}}$. Based on this idea two algorithms have been developed; ACA with full pivoting which is an $O(K^2 \cdot N \cdot L)$ algorithm and requires as starting point the calculation of the entire matrix $\tilde{\mathbf{M}}$, and the partially pivoted ACA, which is an $O\left(K^2\left(N+L\right)\right)$ algorithm requiring the calculation of only a small part of $\tilde{\mathbf{M}}$. The partially pivoted ACA is faster and consumes less memory than the fully pivoted one, but the approximation accuracy ε is not guaranteed because its stopping criterion is heuristic since the $\left\|\tilde{\mathbf{M}}\right\|_F$ in eqn (10) cannot be calculated exactly. In the present work the above mentioned drawback is cancelled applying extra convergence checks. We have seen that for small admissible blocks, where the rank K is comparable with the full rank R, the fully pivoted ACA is more efficient than the partially pivoted one. On the contrary, for large admissible blocks, where K is usually orders of magnitude smaller than R, the partially pivoted ACA is much more efficient. In the present work, full pivoted ACA is used for admissible blocks with full rank R<100.

2.3 Comparison between conventional and Hierarchical ACA/BEM

In order to demonstrate the efficiency of the proposed hierarchical ACA accelerated BEM formulation, the 2D elastostatic bending problem, described in section 2.1, is solved using both ACA and conventional BEM.

In figs 2(a) and (b) the normalized total CPU time and memory requirements as function of DOFs N are depicted, respectively. As total CPU time is considered the time required for the evaluation of the matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$ as well as the time spent for the system solution. The total CPU time is normalized by the corresponding time required for the solution of the problem for 100000 DOFs by means of the conventional BEM. Observing figs 2(a) and (b), one can say that the $O(N^2)$ memory demand in conventional, is verified.



(a) (b)

Figure4. Total CPU time (a) and the memory demand (b), using conventional BEM and Hierarchical ACA/BEM

The CPU time and memory requirements for solving a 2D elastic problem with 10^5 DOFs through conventional BEM are about the same with the corresponding ones needed for the same problem with 10^6 DOFs and solved with the aid of partially pivoted ACA.

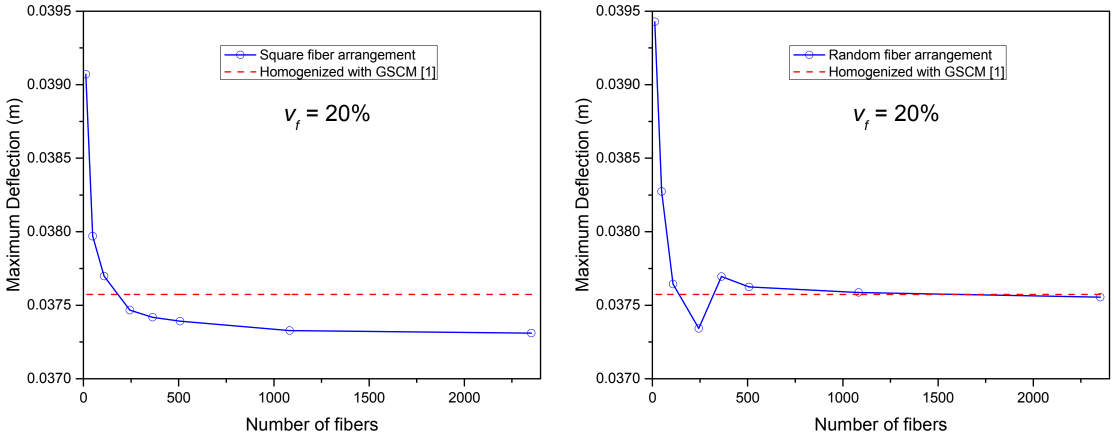
3 NUMERICAL RESULTS

In the present work, the bending problem of the composite plate, described in section 2.1, is solved, by means of the partially pivoted ACA-BEM, for various numbers of fibers and volume fractions u_f, in order to investigate the microstructural effects of the fibers' size with respect to the dimensions of the plate. The results are compared to those obtained by solving the same bending problem, considering the plate homogeneous with effective material properties provided by the GSCM [1]. The problem parameters are listed in the table 1.

Geometry of the plate		Material properties			Bending Load	
		Property	Matrix	Fiber		
Length L (m)	9	Young's modulus E (GPa)	66	360	P (MPa)	10
Width D (m)	3	Poisson's ratio v	0.31	0.25		

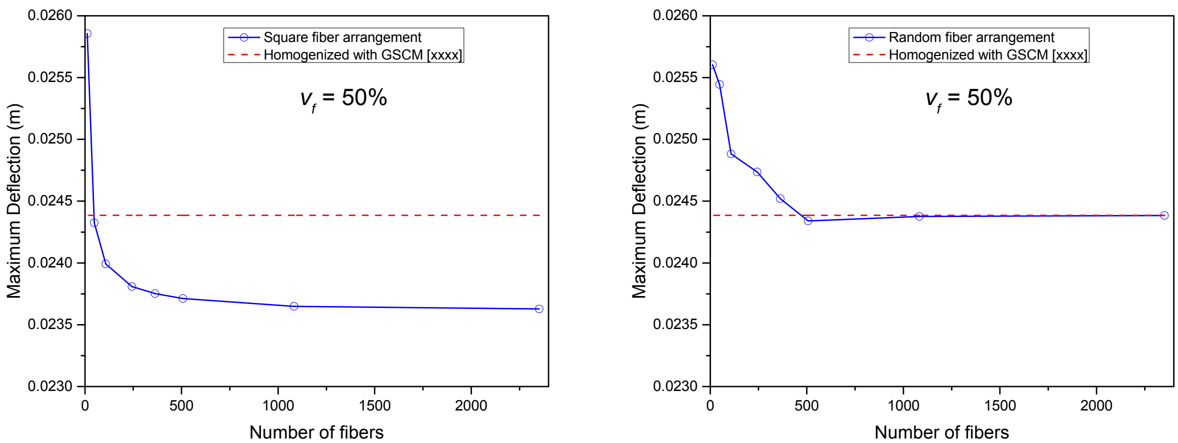
Table 1. Problem parameters

In Figures 3(a,b) and 4(a,b), the maximum deflection of the composite plate with periodically and randomly distributed fibers is depicted for volume fractions 0.20 and 0.50, respectively. Keeping the volume fraction constant, all the problems are solved for different number of fibers, as it is shown in x-axis of both figures.



(a) (b)

Figure 3. Maximum deflection of the composite plate as a function of fibers' radius a, with fibers (a) periodically arranged in a square pattern and (b) randomly distributed, for a volume fraction u_f= 0.20

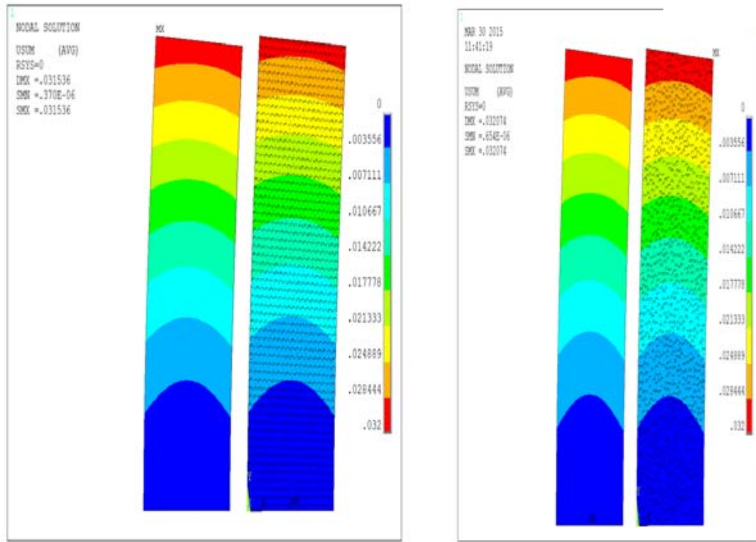


(a) (b)

Figure 4. Maximum deflection of the composite plate as a function of fibers' radius a, with fibers (a) periodically arranged in a square pattern and (b) randomly distributed, for a volume fraction u_f= 0.50

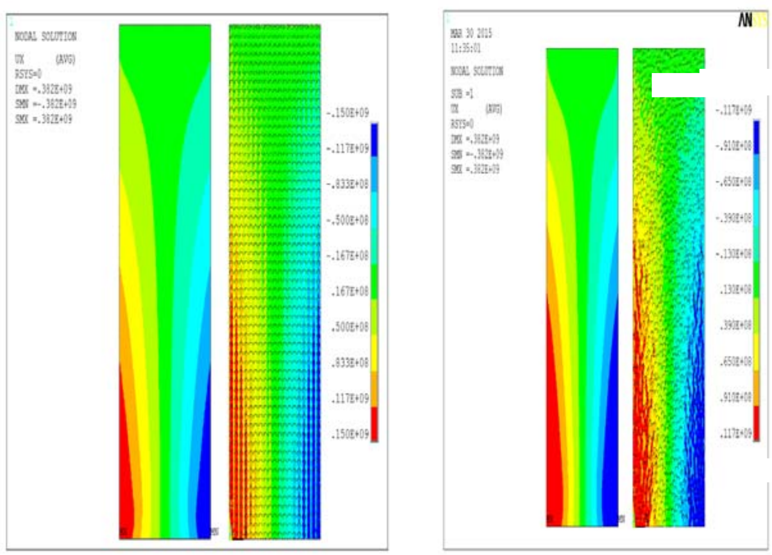
Figures 3 and 4 reveal the following very interesting conclusions: (i) there are significant microstructural effects for a number of fibers less than 500, (ii) as the fibers' size decreases their microstructural effects become less significant, (iii) for a number of fibers greater than 1000 the microstructural effects become negligible, (iv) the GSCM provides accurate predictions only for randomly distributed fibers and for large number of fibers,.(v) the plate with periodically arranged fibers is stiffer than the one-with randomly distributed fibers.

Figure 5 and 6 depict the contours of displacement u_x and normal stress σ_y, respectively for the plates with periodically arranged and randomly distributed fibers with N_f=1083 and u_f=0.35, as well as for the corresponding homogenized plate.



(a) (b) (c)

Figure 5. Contours of the displacement magnitude, for N_f=1083 and u_f =0.35; (a) homogenized plate (b) periodically arranged fibers and (c) randomly distributed fibers.



(a) (b) (c)

Figure 6. Contours of normal bending stress, for N_f=1083 and u_f =0.35; (a) homogenized plate (b) periodically arranged fibers and (c) randomly distributed fibers.

4 CONCLUSIONS

In the present work a large scale elasticity problem concerning the simulation of a fibrous composite plate with a large number of fibers was solved by means of a proposed ACA/BEM. For the simulation, models with up to one million DOFs were used. The proposed methodology reduces the solution time and the memory requirements significantly compared with the corresponding ones needed by the conventional BEM.

The obtained numerical results reveal that there are significant microstructural effects due to the size of the fibers. Also, significant differences were observed in the numerical results between the two examined arrangements of the fibers, i.e., randomly distributed and periodically arranged in a square pattern. The examined micromechanical model GSCM provides accurate predictions for the effective material properties of the composite only for randomly distributed fibers under the constraint that the number of fibers must be large enough in order to the microstructural effects become negligible.

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