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STATIC AND STABILITY ANALYSIS OF GRADIENT ELASTIC BEAMS BY FINITE ELEMENTS

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Abstract. S*tatic and stability stiffness matrices of a gradient elastic flexural Bernoulli-Euler beam finite element are analytically constructed with the aid of the basic and governing equations of equilibrium for that element. The flexural element has two nodes with three degrees of freedom per node, i.e., the displacement, the slope and the curvature. The stability stiffness matrix incorporates the effect of axial compressive force on bending. Use of these stiffness matrices for a plane system of beams enables one through a finite element analysis to determine its response to static loading and its buckling load. Because the exact solution of the governing equation of the problem is used as the displacement function, the resulting stiffness matrices and the obtained structural responses are also exact. Three examples are presented to illustrate the method and show its advantages.*

1. INTRODUCTION

Nanomechanical and nanoelectronic devices, usually in the form of beams, plates and shells, are characterized by extremely small dimensions, which are comparable to their microstructural internal lengths. In these cases, microstructural effects are important and have to be taken into account. Classical theory of elasticity is not capable of taking them into account and one has to use generalized or higher order theories of elasticity, which due to their nonlocal character and the inclusion of internal length parameters can effectively model microstructural effects macroscopically. Among those theories, Mindlin's [1] general theory of elasticity with microstructure in its various forms and simplifications, especially the one with just one internal length parameter commonly known as gradient elasticity theory, has found many applications in various static and dynamic problems [2].

For the particular case of the static and dynamic analysis of flexural gradient elastic beams, the structures considered in this paper, one can mention the works of Papargyri-Beskou et al [3,4], Lam et al [5], Giannakopoulos and Stamoulis [6], Kong at al [7] and Papargyri –Beskou and Beskos [8] on gradient elastic Bernoulli-Euler beams and Papargyri-Beskou et al [9], Wang at al [10], Akgoz and Civalek [11] and Triantafyllou and Giannakopoulos [12] on gradient elastic Timoshenko beams. In all these works, the analysis was done by analytic methods and the beams were simple, statically determinate and under simple type of loading.

However, for statically indeterminate beams with complex type of loading and/or variable cross section and especially for structures composed of beams, analytic methods become complicated, inefficient and hence impractical. For all the above cases, resort should be made to numerical methods of solution, such as the finite element method [13]. Triantafyllou and Giannakopoulos [12] recognizing the problem, tried to provide analysis aids for simple gradient elastic statically determinate and indeterminate Timoshenko beams but they did not develop any numerical method of solution.

In this work, the static and stability stiffness matrices of a gradient elastic flexural Bernoulli-Euler beam finite element are analytically constructed with the aid of the basic and governing equations of equilibrium of that element and its associated possible boundary conditions as described in Papargyri-Beskou et al [3]. Use of these stiffness matrices for a plane system of beams enables one through a finite element analysis to determine its response to static loading and its buckling load. Because the exact solution of the governing equation of the problem is used as the displacement function, the resulting stiffness matrices and the obtained structural response are also exact. Asiminas and Koumousis [14] have recently constructed stiffness matrices for finite element analysis of gradient elastic Bernoulli-Euler beam structures on the basis of the theory in [3]. However, their displacement function is the usual cubic polynomial solution of the classical beam bending problem and as a result of that the resulting matrices are approximate. This implies that every physical member of the structure has to be discretized into a number of 2-3 finite elements for acceptable accuracy solutions. In contrast, because the stiffness matrices of the present work are exact, one can assign only one finite element per physical member and still obtain the exact solution. Of course, the expressions of the stiffness coefficients here are more complicated than those in [14] but they are in closed form and hence the additional computational effort is small.

Three examples are presented to illustrate the method and demonstrate its merits. Two examples deal with static analysis of statically determinate and indeterminate beams and one example with the determination of the buckling load of a beam.

2. GRADIENT ELASTIC BEAM THEORY

The basic and governing equations of a gradient elastic Bernoulli-Euler beam in bending under dynamic lateral loading as well as the associated classical and nonclassical boundary conditions derived in Papargyri-Beskou et al [3] are reproduced in this section for reasons of completeness and easy reference.

Consider a straight prismatic beam under a static lateral load q(x) distributed along the longitudinal axis of the beam, as shown in Fig. 1. Thus, the loading plane is x-y and the cross-section A of the beam is characterized by the two axes y and z with the former one being its axis of symmetry. Under the lateral load q(x), the beam axhibits lateral deflection υ(x) due to bending along the x axis and in the x-y plane. Assuming gradient elastic material behavior, one has that the normal to the cross-section

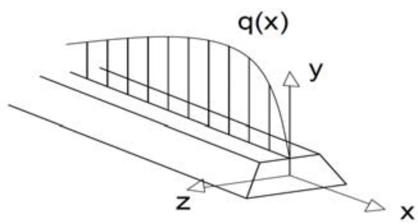

Fig. 1 Geometry and loading of a prismatic beam in bending

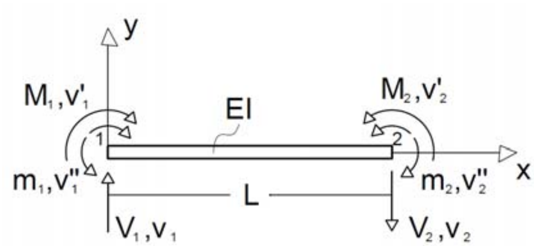

Fig. 2 Mechanics convention for generalized nodal forces and displacements of a gradient elastic flexural beam element.

bending stress σ_x has the form

$$\sigma_x = E(\varepsilon_x - g^2 \frac{\partial^2 \varepsilon_x}{\partial x^2}) \tag{1}$$

where E is the modulus of elasticity, g is the gradient coefficient with dimensions of length (internal length representing the microstructural effects macroscopically) and ε_x is the normal bending strain expressed as

$$\varepsilon_x = -yk = -y\frac{\partial^2 \upsilon}{\partial x^2} \tag{2}$$

Utilizing the kinematics of the Bernoulli-Euler theory, the constitutive relation (1) and the equilibrium of axial forces and bending moments, one can finally obtain the governing equations of equilibrium for the gradient elastic beam in terms of the lateral deflection υ(x) as [3]

$$EI(\frac{\partial^4 \upsilon}{\partial x^4} - g^2\frac{d^6 \upsilon}{dx^6}) = -q(\mathrm{x}) \tag{3}$$

where I is the cross-sectional moment of inertia about the z axis. The above equation, which is of the sixth degree, reduces to the classical one of the fourth degree for g=0.

If one considers a beam element of length L with its two ends defined by x=0 and x=L, as shown in Fig.2, and makes use of a variational statement, he can recover the governing equation (3) and all possible classical and non-classical boundary conditions so as to satisfy the following equations [3]:

$$\begin{aligned}
&[V(L) - EI[\upsilon'''(L) - g^2\upsilon^V(L)]]\delta\upsilon(L) - [V(0) - EI[\upsilon'''(0) - g^2\upsilon^V(0)]]\delta\upsilon(0) = 0 \\
&[M(L) - EI[\upsilon''(L) - g^2\upsilon^{IV}(L)]]\delta\upsilon'(L) - [M(0) - EI[\upsilon''(0) - g^2\upsilon^{IV}(0)]]\delta\upsilon'(0) = 0 \\
&[m(L) - EI\upsilon'''(L)]\delta\upsilon''(L) - [m(0) - EIg^2\upsilon'''(0)]\delta\upsilon''(0) = 0
\end{aligned} \tag{4}$$

In the above, primes denote derivatives with respect to x , V is the shear force, M is the bending moment and m is the double moment due to the microstructure. This double moment m consists of two self-equilibrating moment vectors that do not contribute to the equilibrium equations but to the strain energy, as depicted in Fig.2, where the positive directions of all forces and moments are also shown. It is observed that for g=0, Eq.(4) reduces to the corresponding ones for the classical case. The term classical boundary conditions for the first two of Eqs. (4), used in this work and in most works of the pertinent literature, may not correct in the sense that these two equations contain not only classical but also nonclassical terms as well. However, for g=0, these equations reduce to the classical ones, while the third of Eqs. (4), consisting of only nonclassical terms, disappears.

Consider now the previous beam of Figs 1 and 2 without the lateral load q(x) subjected to an axial compressive constant load P, which can cause flexural buckling for a certain value P_{cr} called the critical load or buckling load to be determined.

The governing equation of a beam in buckling as well as all possible boundary conditions can be obtained with the aid of a variational statement as described in detail in [3]. Thus, the governing equation of a beam in buckling is of the form

$$EI(\upsilon^{IV} - g^2\upsilon^{VI}) + \mathrm{P}\upsilon'' = 0 \tag{5}$$

while the boundary conditions satisfy the equations

$$\begin{aligned}
&[V(L) - [\mathrm{P}\upsilon'(L) + \mathrm{EI}[\upsilon'''(\mathrm{L}) - g^2\upsilon^V(L)]]]\delta\upsilon(L) - [V(0) - [\mathrm{P}\upsilon'(0) + \mathrm{EI}[\upsilon'''(0) - g^2\upsilon^V(0)]]]\delta\upsilon(0) = 0 \\
&[M(L) - EI[\upsilon''(L) - g^2\upsilon^{IV}(L)]]\delta\upsilon'(L) - [M(0) - EI[\upsilon''(0) - g^2\upsilon^{IV}(0)]]\delta\upsilon'(0) = 0 \\
&[m(L) - EIg^2\upsilon'''(L)]\delta\upsilon''(L) - [m(0) - EIg^2\upsilon'''(0)]\delta\upsilon''(0) = 0
\end{aligned} \tag{6}$$

It is easy to see that if g = 0 in Eqs (5) and (6), the classical versions of these equations are recovered.

3. GRADIENT ELASTIC FLEXURAL STIFFNESS MATRIX

This section deals with the development of the stiffness matrix of a gradient elastic flexural beam element with two nodes 1 and 2 at its two ends, as shown in Fig.3. On the basis of Eqs (4) describing the boundary conditions of the problem, one concludes that there are three nodal generalized displacements (υ, υ`, υ``) and three nodal generalized forces (V, M, m) associated with those displacements at every node, as shown in Fig. 3. In other words, every node has three degrees of freedom (d.o.f.): displacement, slope and curvature. These nodal generalized displacements and forces of Fig. 3 are considered to be positive. One can observe that the positive nodal quantities of Fig. 3 (matrix convention) are not the same with the corresponding ones of Fig. 2 (mechanics convention) and this fact has to be taken into account when transferring information from one figure to the other.

For the construction of the stiffness matrix of the finite element of Fig.3 one needs to select a displacement function and adopt a definition of that matrix. In this work, the displacement function is selected to be the exact solution of the homogeneous part of Eq. (3), which has the form [3]

$$\upsilon(\mathrm{x}) = \mathrm{C}_1 x^3 + C_2 x^2 + C_3 x - + C_4 + C_5 g^4 \sinh(\mathrm{x}/\mathrm{g}) + \mathrm{C}_6\, g^4 \cosh(\mathrm{x}/\mathrm{g}) \tag{7}$$

where C_1, C_2,…,C_6 are constants of integration to be determined. It is observed that in the limit as g->0, the above displacement function reduces to the cubic polynomial used in classical finite element analysis. Because use is made of the exact solution of the governing equation of the problem as the displacement function, it is expected that the stiffness matrix to be constructed will be exact and hence the response of a beam structure to static loading analyzed on the basis of the finite element method with that element stiffness matrix for every physical member will be also exact.

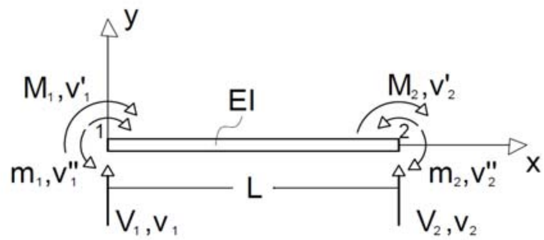

Fig.3. Matrix convention for generalized nodal forces and displacements of a gradient elastic flexural beam element

The stiffness matrix of the finite element of Fig. 3 will be constructed here on the basis of the displacement function (7) and the basic definition of any coefficient of that matrix. Thus, for a stiffness matrix [K] connecting the vector of generalized nodal forces {F} with the vector {U} of the corresponding nodal displacements as

$$\{F\} = [K]\ \{U\} \tag{8}$$

the stiffness coefficient K_{ij} is defined as the nodal generalized force at the degree of freedom i due to unit nodal generalized displacement of the degree of freedom j, while all the other displacements are zero. In this work, since every node has 3 dof, the finite element of Fig. 3 has 2x3=6 dof and hence the size of the stiffness matrix [K] will be 6x6, while the indices ı, j will take the values 1, 2, …, 6. Thus, the stiffness matrix will be onstructed here column by column on the basis of six generalized displacements states, the displacement function (7) and the expressions for V, M, m in terms of that displacement and its derivatives as defined in Eqs (4).

Thus the derivatives of $\upsilon = \upsilon(\mathrm{x})$ of Eq. (7) are evaluated and listed as

$$\begin{aligned}
\upsilon^{'}(\mathrm{x}) &= 3C_1x^2 + 2C_2x + C_3 + C_5g^3\cosh(\mathrm{x}/\mathrm{g}) + \mathrm{C}_6\, g^3\sinh(\mathrm{x}/\mathrm{g}) \\
\upsilon^{''}(\mathrm{x}) &= 6C_1x + 2C_2 + C_5g^2\sinh(\mathrm{x}/\mathrm{g}) + \mathrm{C}_6\, g^2\cosh(\mathrm{x}/\mathrm{g}) \\
\upsilon^{'''}(\mathrm{x}) &= 6C_1 + C_5g\cosh(\mathrm{x}/\mathrm{g}) + \mathrm{C}_6\, g\sinh(\mathrm{x}/\mathrm{g}) \\
\upsilon^{\mathrm{IV}}(\mathrm{x}) &= C_5\sinh(\mathrm{x}/\mathrm{g}) + \mathrm{C}_6\cosh(\mathrm{x}/\mathrm{g}) \\
\upsilon^{\mathrm{V}}(\mathrm{x}) &= C_5(1/\mathrm{g})\cosh(\mathrm{x}/\mathrm{g}) + \mathrm{C}_6(1/\mathrm{g})\sinh(\mathrm{x}/\mathrm{g})
\end{aligned} \tag{9}$$

while the generalized forces V, M and m are expressed in terms of the above derivatives with the aid of Eqs (4) as

$$\begin{aligned}
V(\mathrm{x}) &= EI(\upsilon^{'''} - \mathrm{g}^2\,\upsilon^{V}) \\
\mathrm{M}(\mathrm{x}) &= EI(\upsilon^{''} - \mathrm{g}^2\,\upsilon^{IV}) \\
m(\mathrm{x}) &= EIg^2\upsilon^{'''}
\end{aligned} \tag{10}$$

Consider the first displacement state defined as

$$\begin{aligned}
&\upsilon(0) = \upsilon_1 = 1, \qquad \upsilon^{'}(0) = \upsilon_1^{'} = 0, \qquad \upsilon^{''}(0) = \upsilon_1^{''} = 0\,, \\
&\upsilon(\mathrm{L}) = \upsilon_2 = 0, \qquad \upsilon^{'}(\mathrm{L}) = \upsilon_2^{'} = 0, \qquad \upsilon^{''}(\mathrm{L}) = \upsilon_2^{''} = 0\,,
\end{aligned} \tag{11}$$

where $\upsilon_1, \upsilon_1^{'}, \upsilon_1^{''}, \upsilon_2, \upsilon_2^{'}, \upsilon_2^{''}$, are the generalized nodal displacements of the finite element of Fig. 3. The above Eqs (11) in view of the expressions (9) can be written in the form

$$\begin{aligned}
&C_4 + C_6\, g^4 = 1 \ , \quad C_3 + C_5\, g^3 = 0 \ , \quad 2\, C_2 + C_6\, g^2 = 0 \\
&C_1\, L^3 + C_2\, L^2 + C_3\, L + C_4 + C_5\, g^4 \sinh(L/g) + C_6\, g^4 \cosh(L/g) = 0 \\
&3\, C_1\, L^2 + 2\, C_2\, L + C_3 + C_5\, g^3 \cosh(L/g) + C_6\, g^3 \sinh(L/g) = 0 \\
&6\, C_1\, L + 2\, C_2 + C_5\, g^2 \sinh(L/g) + C_6\, g^2 \cosh(L/g) = 0
\end{aligned} \tag{12}$$

The above Eqs (12) can be thought of as a system of six equations with six unknowns, the constants C_i (i=1, 2, …, 6) and easily solved to provide those constants

Using the definition of the stiffness coefficients K_{ij} and the different sign convention of Fig. 2 (mechanics convention) and Fig. 3 (matrix convention), one can obtain the K_{ij} stiffness coefficients for the first column of [K] corresponding to displacement state (11) in the form

$$\begin{aligned}
&K_{11} = V(0),\ K_{21} = M(0),\ K_{31} = m(0) \\
&K_{41} = -V(L),\ K_{51} = -M(L),\ K_{61} = -m(L)
\end{aligned} \tag{13}$$

where the right hand sides of Eqs (13) can be computed by using Eqs (10) and (9) with values for the constants C_i (i=1, 2, …, 6) those obtained by solving Eq. (12). Thus, the above stiffness coefficients K_{ij} of Eq. (8) can be written explicitly in the form:

$$
\begin{aligned}
K_{11} &= 12EI / (12\,g^2 L + L^3 - 6\,gL^2 \coth(L/(2\,g))) \\
K_{21} &= -6EI / (12\,g^2 + L^2 - 6\,gL\coth(L/(2\,g))) \\
K_{31} &= EI / ((1/L) - (L/(12g^2 + L^2 - 6\,gL^2\coth(L/(2\,g))) \\
K_{41} &= -(12EI / (12\,g^2 L + L^3 - 6\,gL^2\coth(L/2\,g))) \\
K_{51} &= -6EI / (12\,g^2 + L^2 - 6\,gL\coth(L/(2\,g))) \\
K_{61} &= EI / ((-1/L) + (L/(12g^2 + L^2 - 6\,gL^2\coth(L/(2\,g)))
\end{aligned}
\qquad (14)
$$

The above procedure is repeated in exactly the same way for the other five remaining displacement states identified by $\upsilon^{'}(0) = \upsilon_1^{'} = 1, \upsilon^{"}(0) = \upsilon_1^{"} = 1, \upsilon(L) = \upsilon_2 = 1, \upsilon^{'}(L) = \upsilon_2^{'}(L) = 1, \upsilon^{"}(L) = \upsilon_2^{"} = 1,$
and the five remaining columns of the stiffness matrix [K] are explicitly derived in closed form. Due to the symmetry of the matrix [K], the satisfaction of the relation $K_{ij} = K_{ji}$ serves as a verification of the exactness of these expressions for K_{ij}. Thus, the stiffness equation (8) connecting the vectors {F} and {U} through the stiffness matrix [K] for the finite element of Fig.3 can be explicitly written down with

$$
\begin{aligned}
\{F\} &= \{V_1, M_1, m_1, V_2, M_2, m_2\}^T \\
\{U\} &= \{\upsilon_1, \upsilon_1^{'}, \upsilon_1^{"}, \upsilon_2, \upsilon_2^{'}, \upsilon_2^{"}\}^T
\end{aligned}
\qquad (15)
$$

and the various elements K_{ij} of the matrix [K] given explicitly in Pegios et al [15]. One can prove through a limiting process that for g = 0, the expressions for the K_{ij} stiffness coefficients reduce to the classical ones found, e.g., in [13].

4. GRADIENT ELASTIC STABILITY STIFFNESS MATRIX

This section deals with the development of the stiffness matrix of a gradient elastic flexural beam element subjected to a constant compressive axial load P and having two nodes 1 and 2 at its two ends, as shown in Fig.4. The generalized nodal forces and nodal displacements shown in Fig 4 are the same as those in Fig 3 for the simple flexural beam element. However, in this case the axial load P creates second order moments and the governing equation is Eq. (5) instead of Eq. (3).

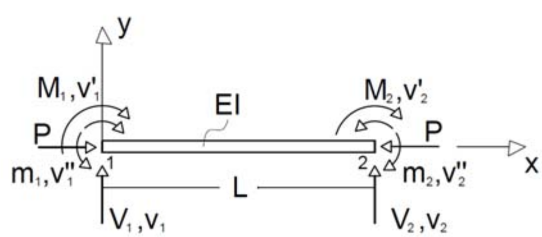

Fig.4. Matrix convention for generalized nodal forces and displacement of gradient elastic flexural beam element with a constant compressive axial force

For the construction of the stability stiffness matrix [S] of the finite element of Fig.4 the displacement function is selected to be the exact solution of Eq.(5), which has the form [3]

$$\upsilon(x) = C_1\,x + C_2 + C_3 \sin\xi x + C_4 \cos\xi x + C_5 \sinh\theta x + C_6 \cosh\theta x \qquad (16)$$

where

$$
\begin{aligned}
\xi &= (1/\sqrt{2g})\sqrt{\sqrt{1+4g^2k^2} - 1} \\
\theta &= (1/\sqrt{2g})\sqrt{1+\sqrt{1+4g^2k^2}} \\
k^2 &= P/EI
\end{aligned}
\qquad (17)
$$

and C_1, C_2,…,C_6 are constants of integration to be determined. It is observed that in the limit as P -> 0, the above displacement function reduces to that of Eq.(7) and as
P -> 0 and g -> 0 to the classical cubic polynomial. Because use is made of the exact solution of the governing equation of the problem as the displacement function, as in the previous section, it is expected that the matrix [S] to be exact and the buckling load of the beam structure obtained by using this matrix to be also exact. Following exactly the same procedure as in the previous section, one can obtain the stability stiffness equation

$$\{F\} = [S]\,\{U\} \tag{18}$$

where the vectors {F} and {U} are given again by Eq.(15), while the elements S_{ij} of the stability stiffness matrix [S] are given explicitly in Pegios et al [15]. It should be noticed that all the entries of matrix [S] are complicated functions of P, which in the limit, as P→ 0, reduce to the flexural stiffness coefficients K_{ij} [15]. In case the axial load P is tensile, one can use the aforementioned coefficients S_{ij} but with –P instead of P.

5. NUMERICAL EXAMPLES

Three examples are presented in this section to illustrate the use of the previously developed stiffness matrices in the framework of the finite element method and demonstrate the advantages of the whole procedure. Two examples deal with the analysis of statically determinate and indeterminate beams and one example with determination of the buckling load of a beam.

5.1 Example 1. Consider a simply supported gradient elastic beam of length 2L and flexural rigidity EI under a lateral concentrated vertical load P at the middle of its span, as shown in Fig. 5. The above beam is considered as beam structure composed of two finite elements 1-2 and 2-3, as shown in Fig. 5. With 3 d.o.f. per node, the whole beam has 3 x 3 = 9 d.o.f. and hence the total structural stiffness matrix will be of the order of 9 x 9. This stiffness matrix is obtained by a simple superposition of the stiffness matrices 6 x 6 of the finite elements 1-2 and 2-3 as given by Eq. (8) by following standard procedures [13].

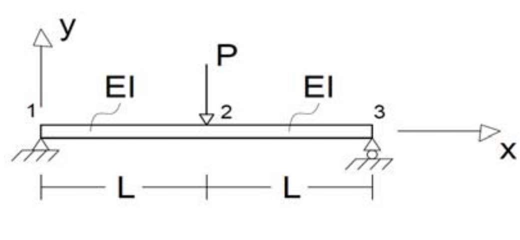

Fig.5 Geometry and loading of a simply supported gradient elastic flexural beam

The classical boundary conditions of the problem in terms of generalized nodal displacements are

$$\upsilon_1 = \upsilon_3 = \upsilon_2' = 0 \tag{19}$$

where the third boundary condition in (19) comes from symmetry considerations. The non-classical boundary conditions are assumed to be

$$\upsilon_1'' = \upsilon_3'' = 0 \tag{20}$$

Thus, the total structural stiffness matrix of order 9 x 9, due to the 5 boundary conditions (19) and (20) becomes 4 x 4 and the stiffness equation takes the form (8) with

$$\{F\} = \{0, -P, 0, 0\}^T$$

$$\{U\} = \{\upsilon_1', \upsilon_2, \upsilon_2'', \upsilon_3'\}^T \tag{21}$$

The above system of 4 equations with 4 unknowns is solved analytically with the aid of Mathematica [16]. Thus, for example, the mid-span deflection υ_2 is given by

$$\upsilon_2 = (-P/6EI)(-3g^2L + L^3 + 3g^3\tanh(L/g)) \tag{22}$$

Fig. 6. Depicts the normalized deflection $\upsilon_2 / \upsilon_2^c$, where $\upsilon_2^c = -PL^3/6EI$ is the classical one, versus g/L.

From this figure, one can easily observe the decrease of the deflection υ_2 for increasing values of the gradient coefficient g (stiffening effect in gradient elasticity).

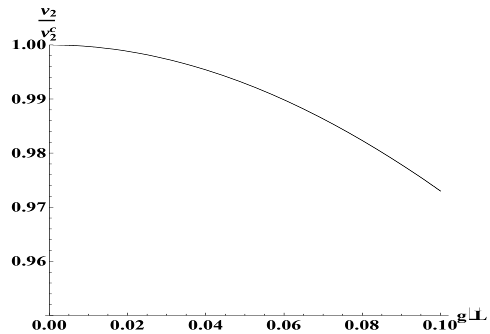

Fig. 6. Deflection versus g/L of the simply supported gradient elastic beam of Fig.5

5.2. Example 2. Consider a gradient elastic beam structure composed of two members with different cross-sections fixed at one end and on rollers at the other end and subjected to a vertical lateral concentrated load P at the middle of its span, as shown in Fig. 7. The flexural rigidities of the two members I and II of Fig. 7 are EI_1 and EI_2, respectively, with $I_1 = 2I_2 = 2I$, while their lengths are equal to L. This structure is modeled by two finite elements 1-2 and 2-3 and has 3 nodes with 3 d.o.f. per node, as shown in Fig. 7. Thus, the total structural stiffness matrix resulting from the superposition of the stiffness matrices of its two elements is 9 x 9. The classical boundary conditions of this problem in terms of generalized nodal displacements are

$$\upsilon_1 = \upsilon_1^{'} = \upsilon_3 = 0 \tag{23}$$

while it is assumed only one non-classical boundary condition of the form

$$\upsilon_1^{''} = 0 \tag{24}$$

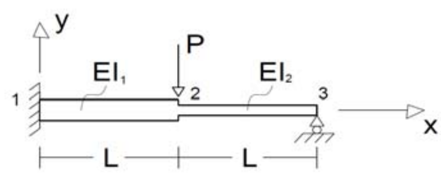

Fig.7. Geometry and loading of a fixed-hinged gradient elastic beam composed of two members with different cross-sections

Application of these 4 boundary conditions in the stiffness equation for the whole structure results in a reduced in size stiffness equation, i.e., in a system of 5 equations with 5 unknown generalized displacements $\upsilon_2, \upsilon_2^{''}, \upsilon_2^{''}, \upsilon_3^{'}, \upsilon_3^{''}$ and a generalized force vector $\{F\} = \{-P, 0, 0, 0, 0\}^T$.

This system of equations is solved analytically with the aid of Mathematica [16] and yields the unknowns $\upsilon_2, \upsilon_2^{'}, \upsilon_2^{''}, \upsilon_3^{'}, \upsilon_3^{''}$. For example,

$$\begin{aligned}
\upsilon_2 = {} & (\mathrm{P}(144\,g^6 - 504\,g^4 L^2 - 96\,g^2 L^2(-6\,g^2 + 5\,L^2)\cosh(L/g) \\
& -3(48\,g^6 - 264\,g^4 L^2 + 16 g^2 L^4 + 11 L^6)\cosh(2\,L/g) \\
& +24 gL(-(-60 g^4 + 8 g^2 L^2 + L^4)\sinh(L/g) + (-18\,g^4 - 21\,g^2 L^2) \\
& +5L^4)\sinh(2L/g)))) / (72\,\mathrm{EI}(-8\,g^2\, L\cosh(L/g) + 9\,L(g^2 \\
& +L^2)\cosh(2\,L/g) + L(-13\,g^2 + 3L^2 - 8gL\sinh(L/g)) + g(6\,g^2 \\
& -13L^2)\sinh(2\,L/g))))
\end{aligned} \tag{25}$$

Figure 8. depicts the normalized deflection $\upsilon_2 / \upsilon_2^c$, where $\upsilon_2^c = -11PL^3 / 216\,\mathrm{EI}$ is the classical one [15], versus g/L. From this figure, one can easily observe the decrease of the deflection υ_2 for increasing values of the gradient coefficient g (stiffening effect in gradient elasticity).

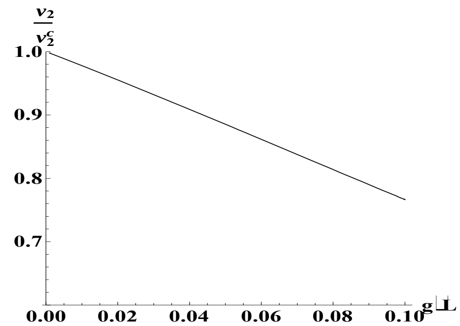

Fig.8. Deflection versus distance for various values of g/L for the beam of Fig.7.

5.3. Example 3. Consider a simply supported gradient elastic uniform beam under the action of an axial compressive force P, as shown in Fig. 9. The beam has length L and flexural rigidity EI. Use will be made of the finite element method to determine the critical or buckling load P_{cr}.

The beam of Fig. 9 is considered to be one finite element with nodes 1 and 2 and 3 d.o.f. per node. Thus, the structural stiffness matrix [S] of this beam is of order 6 x 6 and connects generalized nodal forces to nodal displacements by Eq. (18). The classical boundary conditions in terms of displacements are

$$\upsilon_1 = \upsilon_2 = 0 \tag{25}$$

while the non-classical ones in terms of generalized displacements are assumed to be

$$\upsilon_1'' = \upsilon_2'' = 0 \tag{26}$$

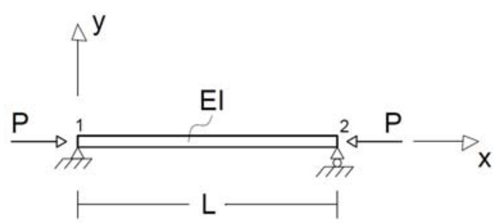

Fig.9. Geometry of a simply supported gradient elastic beam under an axial compressive load

Application of the above 4 boundary conditions reduces Eq.(18) to

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} S_{22} & S_{25} \\ S_{52} & S_{55} \end{bmatrix} \begin{Bmatrix} \upsilon_1' \\ \upsilon_2' \end{Bmatrix} \tag{27}$$

where the stability stiffness coefficients S_{22}, $S_{25} = S_{52}$, S_{55} are given explicitly in Pegios at al [15].
The critical load P_{cr} is found by solving the determinant equation

$$S_{22}\, S_{55} - S_{25}^2 = 0 \tag{28}$$

for P and keeping the first root of the solution. Indeed substitution in (28) of the coefficients S_{22}, S_{55} and S_{25} with their expressions in Pegios et al [15] yields

$$P_{cr} = \frac{\pi^2 EI}{L^2}\left(1 + \pi^2 \left(\frac{g}{L}\right)^2\right) \tag{29}$$

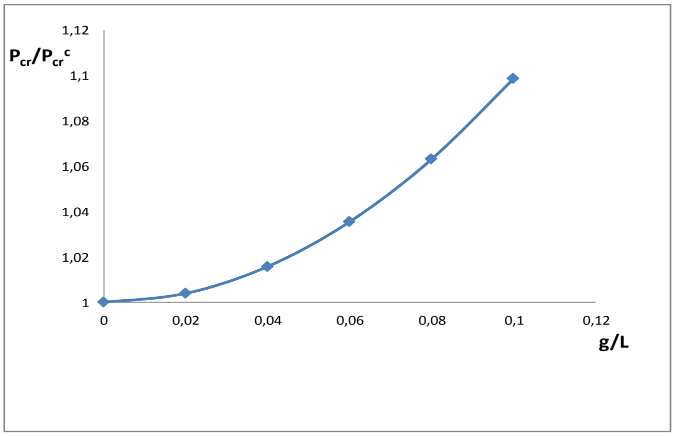

Fig.10. Variation of the dimensionless buckling load P_{cr}/P_{cr}^c versus g/L for the fig. 9..

which is exactly the same with that found in [3] analytically. Fig.10 depicts the variation of P_{cr}/P_{cr}^c versus g/L, where $P_{cr}^c = \pi^2 EI/L^2$ is the classical value of P_{cr}, and clearly shows an increase of this dimensionless buckling ratio for increasing values of the dimensionless gradient coefficient g/L with a constantly increasing rate of increase.

6. CONCLUSIONS

On the basis of the results presented in the previous sections, the following conclusions can be stated:

1. A finite element for a gradient elastic Bernoulli-Euler beam in bending under static loading has been developed with two nodes (at the ends of the element) and three degrees of freedom per node, the displacement, the slope and the curvature.

2. The displacement function of this formulation was selected to be the exact solution of the governing equation of equilibrium of the beam resulting in the exact element stiffness matrix. As a result, use of this matrix in a finite element analysis provides the exact response of a beam structure to static loading.

3. When the effect of an axial compressive load on bending is taken into account and use is made of the exact solution of the buckling governing equation as the displacement function, the exact stability stiffness matrix of an element is obtained. This leads to the exact value of the critical or buckling load.

4. Three examples are presented to illustrate the method and demonstrate its advantages when applied to problems of static and stability analysis of plane gradient elastic beam structures.

5. The advantages of the finite element method presented here are generality, versatility, easy and systematic way of treatment, computational efficiency and obtaining of the exact structural response or buckling load.

6. It was observed, at least in all the examples consider here, a decrease of deflections and an increase of the buckling loads for increasing values of the gradient coefficient, in agreement with the well-known stiffening effect in gradient elasticity.

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