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LINEAR AND GEOMETRICALLY NONLINEAR ANALYSIS OF IN-PLANE THIN SHALLOW ARCHES

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Abstract. *In this paper an integral equation solution to the linear and geometrically nonlinear analysis of non-uniform in-plane shallow arches is presented. Arches exhibit advantageous behavior over straight beams due to their curvature which increases the overall stiffness of the structure. They can span large areas by resolving forces into mainly compressive stresses and, in turn confining tensile stresses to acceptable limits. Most arches are designed to operate linearly under service loads. However, their slenderness nature makes them susceptible to large deformations especially when the external loads increase beyond the service point. Loss of stability may occur, known also as snap-through buckling, with catastrophic consequences for the structure. Linear analysis cannot predict this type of instability and a geometrically nonlinear analysis is needed to describe efficiently the response of the arch. The aim of this work is to cope with the linear and geometrically nonlinear problem of non-uniform shallow arches. The governing equations of the problem are comprised of two nonlinear coupled partial differential equations in terms of the axial (tangential) and transverse (normal) displacements. Moreover, as the cross-sectional properties of the arch vary along its axis, the resulting coupled differential equations have variable coefficients and are solved using a robust integral equation numerical method in conjunction with the arc-length method. The latter method allows following the nonlinear equilibrium path and overcoming bifurcation and limit (turning) points, which usually appear in the nonlinear response of curved structures like shallow arches and shells. Several arches are analyzed not only to validate our proposed model, but also to investigate the nonlinear response of in-plane thin shallow arches.*

1 INTRODUCTION

Curved beams of constant or variable curvature are encountered quite often in many engineering branches such as civil, mechanical and aeronautical engineering. Among structures applications shallow arches have gain a tremendous attention due to their constructional simplicity. In general, arches exhibit advantageous behavior over straight beams due to their curved centerline which increases the overall stiffness of the structure. They can span large areas by resolving forces into mainly compressive stresses and, in turn confining tensile stresses to acceptable limits. Most arches are designed to operate linearly under service loads. However, their slenderness nature makes them susceptible to large deformations especially when the external loads increase beyond the service point. Loss of stability may occur, known also as snap-through buckling, with catastrophic consequences for the structure. Linear analysis cannot predict this type of instability and a geometrically nonlinear analysis is needed to describe efficiently the response of the arch.

The geometrically nonlinear behavior of shallow uniform arches has been scrutinized by numerous investigators using analytical, experimental, and numerical methods. To begin with, analytical methods have been used by Lo and Conway [1] who studied the instability of circular curved beams subjected to equal but opposite end moments, and Conway and Lo [2] who extended the previous investigation to circular curved beam under central concentrated loading and compared their results with the experimental data of Gjelsvik and Bonder [3]. Recently, Pi et al. [4] and Pi and Bradford [5] obtained analytical solutions for the nonlinear behavior for shallow arches with elastic supports and unequal end restraints, respectively.

In addition to that, the vast majority of publications is devoted to numerical methods and especially the Finite Element Method (FEM) (see e.g. [6-19]). Nevertheless, there are several alternatives to FEM numerical solutions that tackle the problem. Patodi and Buragohain [20] suggested a special form of finite difference scheme for geometrically nonlinear analysis of shallow circular arches while Srpčič and Saje [21] adopted the Finite Difference Method (FDM) for the solution of the large deformation problem of thin curved plane beam of constant initial curvature. The work of Xenidis et al. [22] is another interesting approach to the problem in which a truss model is used for the geometrically nonlinear static and dynamic analysis of a thin shallow arch subject to snap-through. Finally, the Integral Equation Method has also been employed by Miyake et al. [23] who studied the geometrically nonlinear bending problem of elastic inextensible arches and Horibe and Asano [24] who solved the large deflections problem of a shallow arch. However, all the above studies were confined to beams with uniform cross-section. To the authors' knowledge this is the first study that tackles the geometrically nonlinear problem of shallow arches with non-uniform cross-section.

The aim of this work is to cope with the linear and geometrically nonlinear problem of non-uniform shallow arches. The governing equations of the problem are comprised of two nonlinear coupled partial differential equations in terms of the axial (tangential) and transverse (normal) displacements, Moreover, the variable cross-sectional properties of the beam result in governing differential equations with variable coefficients which complicate even more the mathematical problem. The solution of the problem is attained using a robust numerical method based on an integral equation technique known as AEM (Analog Equation Method). The method was first developed for the nonlinear analysis of beams with variable stiffness by Katsikadelis and Tsiatas [25] and was also successfully employed to the geometrically nonlinear problem of non-uniform beams resting on nonlinear elastic foundation [26]. The rationale of the method is based on the replacement of the two coupled nonlinear differential equations with variable coefficients by two uncoupled linear ones pertaining to the axial and transverse deformation of a substitute beam with unit axial and bending stiffness, respectively, subjected to unknown fictitious loads. Furthermore, a nonlinear system of equations is formulated with respect to the unknown fictitious loads, which is solved using the arc-length method [27, 28]. This method allows following the nonlinear equilibrium path and overcoming bifurcation and limit (turning) points, which usually appear in the nonlinear response of curved structures like shallow arches and shells. Several arches are analyzed not only to validate our proposed model, but also to investigate the nonlinear response of in-plane thin shallow arches.

2 PROBLEM FORMULATION

In this section, the fundamental equations governing the linear and geometrically - von Kármán type - nonlinear response of non-uniform shallow arches together with the respective boundary conditions are derived in terms of the displacements. The employed nonlinear model takes into account not only the effect of the transverse shear force in the in-plane equilibrium equations but also the influence of the in-plane tangential displacement in the bending response of the arch.

Let us consider a plane curved beam the cross-sections of which are orthogonal to a plane curve (centroid axis) that belongs to the xz plane. A curvilinear abscissa s spans the beam's centroid axis which is bent in its plane under the combined action of the distributed loads $p_t = p_t(s)$ and $p_n = p_n(s)$ acting in the tangential and normal direction, respectively (see Fig. 1). The beam may have a non-uniform cross-section, that is the axial $EA(s)$ and bending stiffness $EI(s)$ vary due to variable cross-sectional properties, $A = A(s)$, $I = I(s)$, and/or nonhomogeneous linearly elastic material $E = E(s)$. In the following the equilibrium equations in terms of the displacements are derived for (a) nonlinear response and (b) linear response.

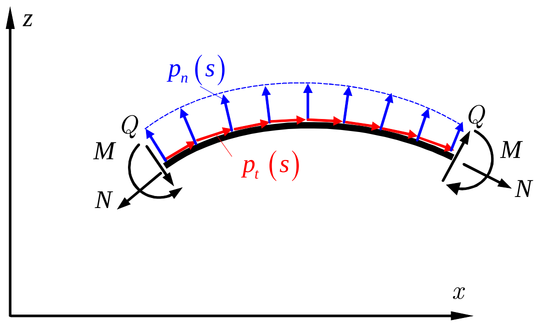


Figure 1. Positive forces and moments acting on a curved element.

2.1 Nonlinear theory

The geometrically nonlinear theory of shallow arches with the von Kármán (or Sanders [29]) nonlinear strain is developed in the same way as in the shallow shell theory. Taking into consideration that the arch is shallow, and without restricting the generality, the nonlinear terms $\left(u/R\right)^2$ and $uw,_s/R$ were omitted [30]. In this case the nonlinear strain-displacement relation is given by

$$\varepsilon_x(s,z)=\varepsilon_0+z\kappa \tag{1}$$

where

$$\varepsilon_0(s,0)=u,_s+\frac{w}{R}+\frac{1}{2}w,_s^2 \tag{2}$$

is the longitudinal and

$$\kappa=-w,_{ss} \tag{3}$$

is the flexural component of strain. Moreover, R is the radius of the arch; $u(s)$ and $w=w(s)$ are the tangential and normal displacements, respectively.

Further, the axial force and the bending moment are integral stress resultants which are given by the relations

$$N=EA\varepsilon_0=EA\left(u,_s+\frac{w}{R}+\frac{1}{2}w,_s^2\right) \tag{4}$$

$$M=EI\kappa=-EIw,_{ss} \tag{5}$$

The governing equations of the curved beam are derived by considering the equilibrium of an elementary section of length ds. In this way, we arrive at the following equations

$$N,_s+\frac{Q}{R}=-p_t \tag{6}$$

$$Q,_s-\frac{N}{R}+\mathbb{N}(u,w)=-p_n \tag{7}$$

$$M,_s-Q=0 \tag{8}$$

where

$$\mathbb{N}=\left(Nw,_s\right),_s \tag{9}$$

is the nonlinear contribution to the equilibrium equations due to the nonlinear strains. Substituting eqn (9) into eqs (6) and (7) and using eqn (8) to eliminate Q, we obtain

$$N,_s+\frac{M,_s}{R}=-p_t\,,\qquad M,_{ss}-\frac{N}{R}+\left(Nw,_s\right),_s=-p_n \tag{10), (11}$$

In order to derive the governing equations in terms of the displacements we substitute eqs (4) and (5) into eqs (10) and (11), thus we have

$$\left[EA\left(u,_s+\frac{w}{R}+\frac{1}{2}w,_s^2\right)\right],_s-\frac{\left(EIw,_{ss}\right),_s}{R}=-p_t \tag{12}$$

$$-\left(EIw,_{ss}\right),_{ss}-\frac{EA}{R}\left(u,_s+\frac{w}{R}+\frac{1}{2}w,_s^2\right)+\left[EA\left(u,_s+\frac{w}{R}+\frac{1}{2}w,_s^2\right)w,_s\right],_s=-p_n \tag{13}$$

The boundary conditions of the problem, that can include elastic support or restraint, are of the form

$$a_1u(0)+a_2N(0)=a_3\,,\quad \bar{a}_1u(l)+\bar{a}_2N(l)=\bar{a}_3 \tag{14), (15}$$

$$\beta_1 w(0)+\beta_2 Q(0)=\beta_3\,,\quad \bar{\beta}_1 w(l)+\bar{\beta}_2 Q(l)=\bar{\beta}_3 \qquad (16),\,(17)$$

$$\gamma_1 w,_s(0)+\gamma_2 M(0)=\gamma_3\,,\quad \bar{\gamma}_1 w,_s(l)+\bar{\gamma}_2 M(l)=\bar{\gamma}_3 \qquad (18),\,(19)$$

where $a_k, \bar{a}_k, \beta_k, \bar{\beta}_k, \gamma_k, \bar{\gamma}_k \; (k=1,2,3)$ are given constants. It is apparent that all types of the conventional boundary conditions (clamped, simply supported etc.) can be derived from eqs (14)-(19) by specifying appropriately these constants.

2.2 Linear theory

Equilibrium equations (10) and (11) after dropping the nonlinear term $Nw,_s$ take the form

$$N,_s+\frac{M,_s}{R}=-p_t\,,\qquad M,_{ss}-\frac{N}{R}=-p_n \qquad (20),\,(21)$$

Further, the bending moment is given by eqn (5), whilst the axial force is derived from eqn (4) as

$$N=EA\left(u,_s+\frac{w}{R}\right) \qquad (22)$$

Finally, by substituting eqs (5) and (22) into eqs (20) and (21) we obtain the equilibrium equations in terms of displacements as follows

$$\left[EA\left(u,_s+\frac{w}{R}\right)\right],_s-\frac{(EIw,_{ss}),_s}{R}=-p_t\,,\qquad -(EIw,_{ss}),_{ss}-\frac{EA}{R}\left(u,_s+\frac{w}{R}\right)=-p_n \qquad (23),\,(24)$$

together with the pertinent boundary conditions given by eqs (14) - (19).

3 THE AEM SOLUTION FOR THE NONLINEAR ANALYSIS OF NON-UNIFORM SHALLOW ARCHES

The boundary value problem described by eqs (12) - (19) is solved using the AEM [25, 26], a robust numerical method based on an integral equation technique. The analog equations for the problem at hand are

$$u,_{ss}=b_1(s)\,,\qquad w,_{ssss}=b_2(s) \qquad (25),\,(26)$$

Eqs (25) and (26) describe the axial and bending linear response of a beam with constant unit axial and flexural stiffness subjected to the unknown fictitious axial b_1 and transverse b_2 loads, respectively. The solution of eqs (25) and (26) at a point $x\in(0,l)$ is obtained in integral form as

$$u(s)=c_1 s+c_2+\int_0^l G_1(s,\xi)b_1(\xi)\,d\xi \qquad (27)$$

$$w(s)=c_3 s^3+c_4 s^2+c_5 s+c_6+\int_0^l G_2(s,\xi)b_2(\xi)\,d\xi \qquad (28)$$

where $c_i \;\; (i=1,2,\ldots 6)$ are arbitrary integration constants to be determined from the boundary conditions and $G_1=\frac{1}{2}|s-\xi|$, $G_2=\frac{1}{12}|s-\xi|(s-\xi)^2$ are the fundamental solutions (free space Green's functions) of eqs (25) and (26), respectively.

The derivatives of u and w are obtained by direct differentiation of eqs (27) and (28). Thus, we have

$$u,_s(s)=c_1+\int_0^l G_1,_s(s,\xi)b_1(\xi)\,d\xi\,,\quad w,_s(s)=3c_3 s^2+2c_4 s+c_5+\int_0^l G_2,_s(s,\xi)b_2(\xi)\,d\xi \qquad (29),\,(30)$$

$$w,_{ss}(s)=6c_3 s+2c_4+\int_0^l G_2,_{ss}(s,\xi)b_2(\xi)\,d\xi\,,\quad w,_{sss}(s)=6c_3+\int_0^l G_2,_{sss}(s,\xi)b_2(\xi)\,d\xi \qquad (31),\,(32)$$

Substituting eqs (27) - (32) into eqs (12) and (13) yields the equations, from which the fictitious sources b_1 and b_2 can be determined. This can be implemented only numerically as follows.

The interval $(0,l)$ is divided into N equal elements on which b_1 and b_2 are assumed constant. After discretization of eqs (27) and (28) we obtain

$$u(s) = \mathbf{H}_1(s)\mathbf{c}_1 + \mathbf{G}_1(s)\mathbf{b}_1 , \qquad w(s) = \mathbf{H}_2(s)\mathbf{c}_2 + \mathbf{G}_2(s)\mathbf{b}_2 \tag{33), (34}$$

where $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ are $1\times N$ known matrices originating from the integration of the kernels $G_1(s,\xi)$ and $G_2(s,\xi)$ on the elements, respectively; $\mathbf{H}_1(s) = \begin{bmatrix} s & 1 \end{bmatrix}$ and $\mathbf{H}_2(s) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix}$; $\mathbf{c}_1 = \{c_1, c_2\}^T$; $\mathbf{c}_2 = \{c_3, c_4, c_5, c_6\}^T$; $\mathbf{b}_1$, $\mathbf{b}_2$ are the vectors containing the values of the fictitious loads at the nodal points, respectively. Similarly, we obtain for eqs (29) - (32)

$$u_{,s}(s) = \mathbf{H}_{1s}(s)\mathbf{c}_1 + \mathbf{G}_{1s}(s)\mathbf{b}_1 , \quad w_{,s}(s) = \mathbf{H}_{2s}(s)\mathbf{c}_2 + \mathbf{G}_{2s}(s)\mathbf{b}_2 \tag{35), (36}$$

$$w_{,ss}(s) = \mathbf{H}_{2ss}(s)\mathbf{c}_2 + \mathbf{G}_{2ss}(s)\mathbf{b}_2 , \quad w_{,sss}(s) = \mathbf{H}_{2sss}(s)\mathbf{c}_2 + \mathbf{G}_{2sss}(s)\mathbf{b}_2 \tag{37), (38}$$

where $\mathbf{G}_{1s}(s)$, $\mathbf{G}_{2s}(s)$, $\mathbf{G}_{2ss}(s)$, $\mathbf{G}_{2sss}(s)$ are $1\times N$ known matrices, originating from the integration of the derivatives of the kernels $G_1(s,\xi)$, $G_2(s,\xi)$ on the elements; $\mathbf{H}_{1s}(s)$ is a 1×2 known matrix resulting from the differentiation of $\mathbf{H}_1(s)$, whereas $\mathbf{H}_{2s}(s)$, $\mathbf{H}_{2ss}(s)$, $\mathbf{H}_{2sss}(s)$ are 1×4 known matrices resulting from the differentiation of $\mathbf{H}_2(s)$.

Finally, collocating eqs (12) and (13) at the N nodal points and substituting the discretized eqs (35) - (38) yields the following equations

$$\mathbf{F}_1(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) = -\mathbf{p}_t , \qquad \mathbf{F}_2(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) = -\mathbf{p}_n \tag{39), (40}$$

where $\mathbf{F}_i(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c})$ are generalized stiffness vectors and $\mathbf{c} = \{c_1, c_2, \ldots c_6\}^T$. Eqs (39) and (40) constitute a system of $2N$ nonlinear algebraic equations with $2N+6$ unknowns. The required six additional equations result from the boundary conditions. Thus, after substituting the relevant derivatives into eqs (14) - (19), we obtain 6 additional nonlinear equations

$$\mathbf{f}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) = \mathbf{0} \tag{41}$$

where $\mathbf{f}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c})$ is a vector which depends nonlinearly on the unknown fictitious loads $\mathbf{b}_1$, $\mathbf{b}_2$ and the arbitrary constants $\mathbf{c}$.

For a certain value of the external loads $\mathbf{p}_n, \mathbf{p}_t$ the nonlinear equations (39) - (41) are solved numerically using Newton-Rapshon iterations to yield $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{c}$. By increasing the external load we can obtain the equilibrium path and investigate the nonlinear response of the structure. The governing equations (39) - (41) can be written as

$$\mathbf{F}(\mathbf{x}, \lambda) = \begin{Bmatrix} \mathbf{F}_1(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) + \lambda \mathbf{p}_t \\ \mathbf{F}_2(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) + \lambda \mathbf{p}_n \\ \mathbf{f}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) \end{Bmatrix} = \mathbf{0} \tag{42}$$

where $\mathbf{x} = \{\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{c}\}^T$ is a vector with the $2N+6$ unknown variables and λ is the load parameter.

The above procedure has limited success when bifurcation points appear in the equilibrium path and fails completely in presence of limit (turning) points. A shallow arch is a characteristic type of structure which becomes unstable at a certain value of the external vertical load and a snap-through buckling (limit point) occurs. In this case continuation methods, such as the arc-length method [27, 28], are able to trace the equilibrium path and overcome successfully bifurcation or limit points.

In the arc-length method the unknown variables $\mathbf{x}$ and the load parameter λ are expressed as a function of the arc-length s of the equilibrium path (load-displacement curve)

$$\mathbf{x} = \mathbf{x}(s) , \qquad \lambda = \lambda(s) \tag{43), (44}$$

Moreover, the nonlinear system of eqs (42) is augmented by adding a constraint equation of the form

$$\left\| \frac{d\mathbf{x}}{ds} \right\|^2 + \left(\frac{d\lambda}{ds} \right)^2 = 1 \tag{45}$$

where $\|\bullet\|$ is the Euclidian norm. The inclusion of the constraint equation (45) ensures that the load parameter λ changes in the direction where a solution of the nonlinear set of eqs (42) is feasible.

In this paper we use the pseudo arc-length method [31], where the additional constraint has the following form

$$\Lambda(\mathbf{x},\lambda)=(\mathbf{x}-\mathbf{x}_{j-1})(\mathbf{x}_{j-1}-\mathbf{x}_{j-2})+(\lambda-\lambda_{j-1})(\lambda_{j-1}-\lambda_{j-2})-\Delta s^2=0 \tag{46}$$

and $\mathbf{x}_{j-1}$ and λ_{j-1} being the values of the unknown variables at the previous step of the Newton-Rapshon iterations and Δs is a parameter of the method. Thus, the $2N+6$ algebraic eqs (42) and the constraint eqn (46) are solved numerically using the Newton-Rapshon method for the $2N+6$ unknown variables $\mathbf{x}$ and the load parameter λ .

4 NUMERICAL EXAMPLES

4.1 Linear shallow arch with uniform cross-section

As a first example we study the linear response of clamped circular shallow arch subjected to a central vertical concentrated load. The arch has a uniform rectangular cross-section $b\times h_0$. The geometry of the arch is shown in Fig. 2. The data employed are $E=10^7\,\text{kN/m}^2$, $\nu=0.3$, $R=133.14m$, $L=34m$, $b=1m$ and $h_0=0.1875m$. The results are obtained using $N=51$ elements and are in very good agreement compared with a FEM solution using 171 beam elements. In Fig. 3 the profiles of both the normal and the tangential displacement are depicted. Moreover, in Fig. 4 the profiles of the bending moment and axial force are also presented.

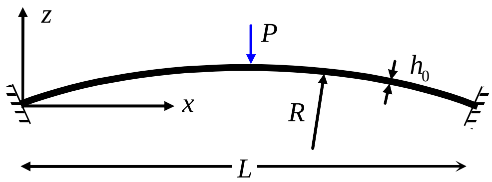


Figure 2. Shallow arch geometry.

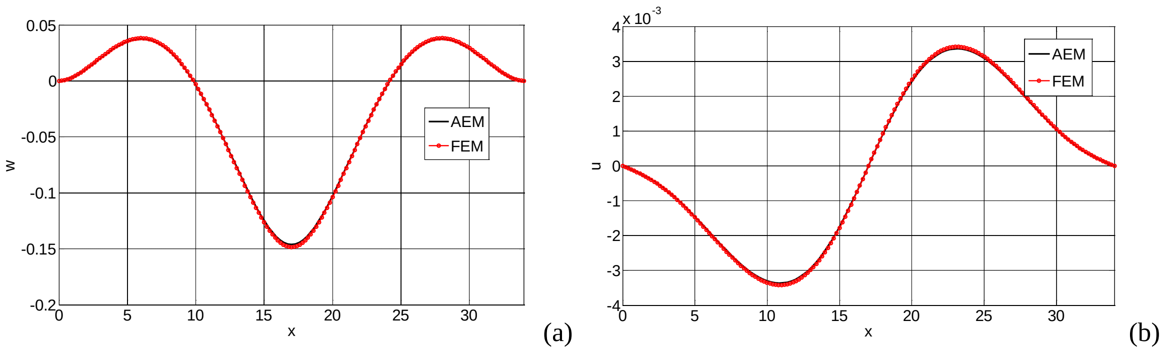


Figure 3. Profiles of the (a) normal and (b) tangential displacement in example 4.1.

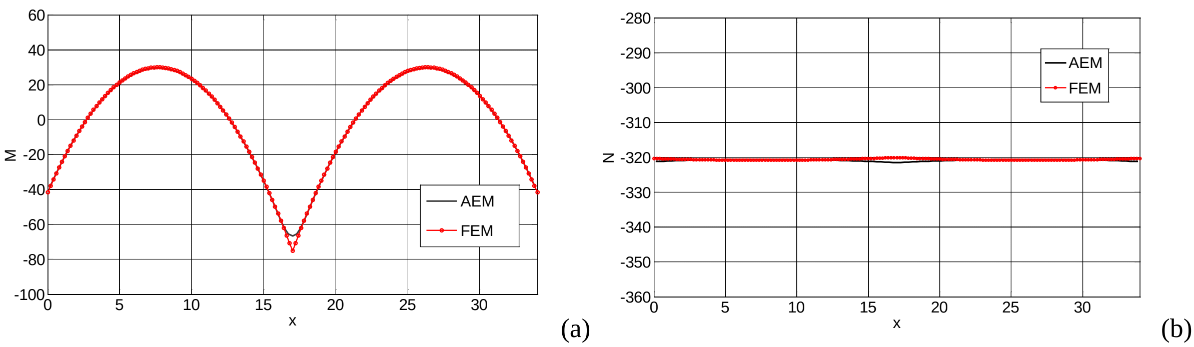


Figure 4. Profiles of the (a) bending moment and (b) axial force in example 4.1.

4.2 Nonlinear shallow arch with uniform cross section

The nonlinear response of the shallow arch of the previous example is studied. The results are obtained with $N = 51$ and $N = 121$ elements. Fig. 5 presents the load versus normal displacement curves at the center of the arch for the linear and nonlinear response. The nonlinear results are compared with that obtained by Surana and Sorem [16] using the FEM. Three parabolic three-dimensional beam elements were employed in [16] and the displacement field approximation was expressed in terms of nodal translations and non-linear functions of nodal rotations. In Figs. 6 and 7 the normal displacement, the bending moment and the axial force are depicted, respectively, for various cases of the external load. It is observed that a snap-through appears when the external load becomes greater than $P_{cr} = 33.65\text{kN}$. Above that critical point the arch becomes unstable and the displacements increase suddenly to large values (see Fig. 6). Moreover, the deformation shape and the stress distribution changes considerably after the critical point.

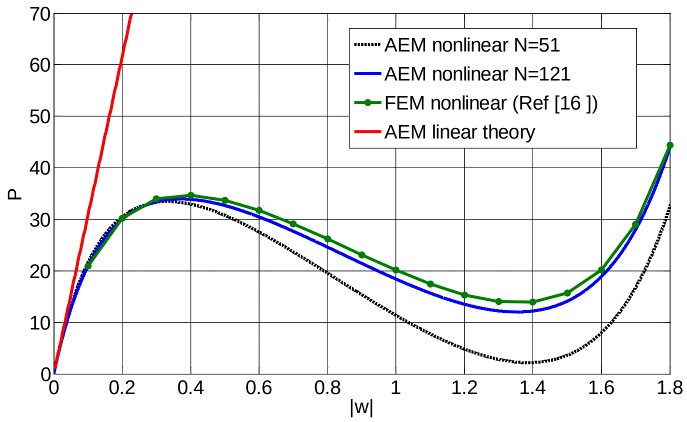


Figure 5. Load versus normal displacement curves at the center of the arch in example 4.2.

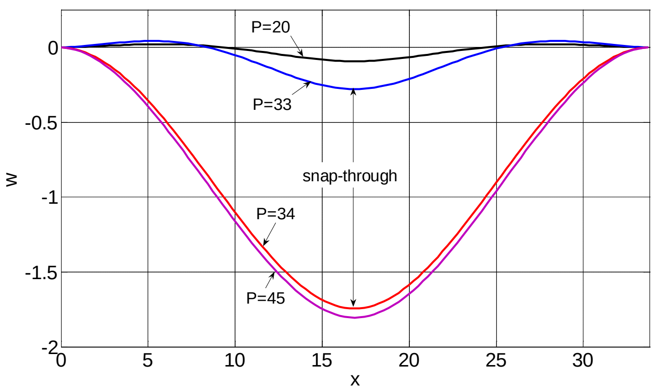


Figure 6. Profile of the normal displacement for various values of the load in example 4.2.

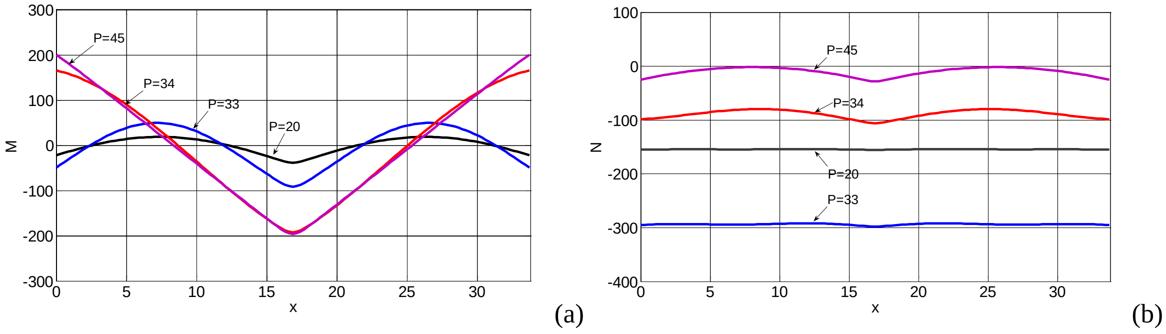


Figure 7. Profiles of the (a) bending moment and (b) axial force for various values of the load in example 4.2.

4.3 Nonlinear shallow arch with non-uniform cross section

The final example is devoted to the clamped circular arch of Fig. 8 with non-uniform rectangular cross-section of constant width $b = 1m$ and parabolically varying height $h(s) = 4h_0\left(s/L\right)^2 - 4h_0\left(s/L\right) + 2h_0$. The data employed are $E = 10^7\,\text{kN/m}^2$, $\nu = 0.3$, $R = 133.14m$, $L = 34m$. The arch is subjected to a central vertical concentrated load. The nonlinear response is studied using $N = 121$, 171 and 201 elements. The results are compared with a FEM solution using 338 beam elements by dividing the arch into 16 segments of constant cross-section. Fig. 9 presents the load versus normal displacement curves at the center of the arch with $h_0 = 0.09375m$. The critical value of the load is $P_{cr} = 6.7\text{kN}$ above which the arch becomes unstable and a snap-through appears. In Figs. 10 and 11 the profiles of the normal and tangential displacement are presented, respectively, for two cases of the external load, before and after the limit point, and are compared with the FEM results. Finally, in Fig. 12 the load-normal displacement curves are presented at the center of the arch with uniform and non-uniform cross-sections. Three cases of height variation are considered, namely, (i) parabolically varying height with $h_0 = 0.09375m$, (ii) uniform height with $h = 0.1875m$ and (ii) parabolically varying height with $h_0 = 0.1875m$. It can be seen from Fig. 12 that the height variation profoundly influences the nonlinear equilibrium path and the limit points of the arch. As a result, by changing the geometric characteristics of the arch cross-section we can alter not only the buckling load (limit point) but also the post-buckling (snap-through) response of the arch.

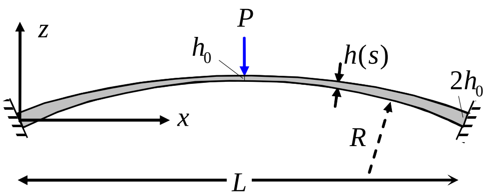


Figure 8. Shallow arch geometry in example 4.3.

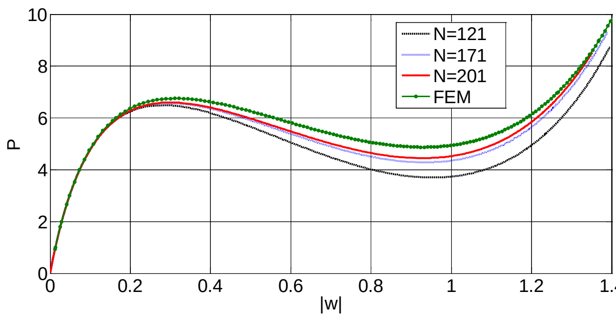
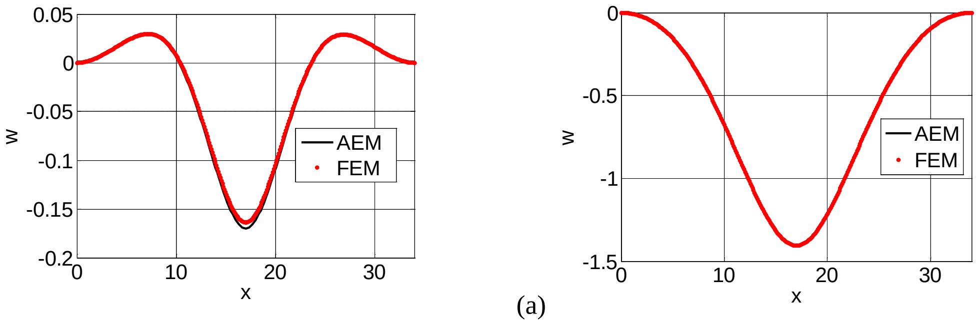
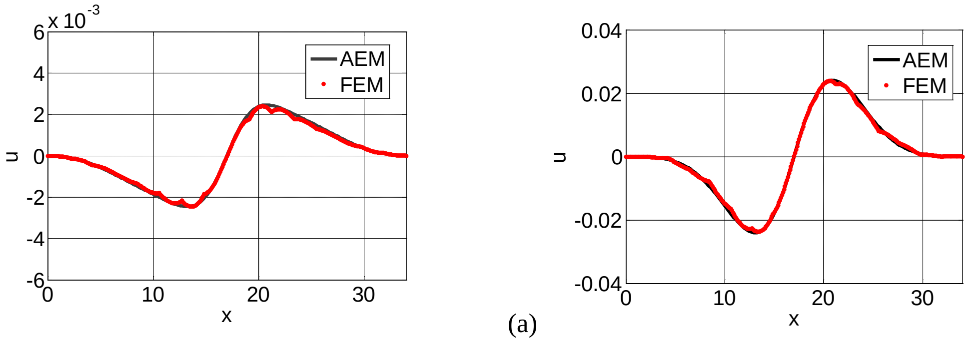


Figure 9. Load versus normal displacement curves at the center of the arch in example 4.3 ($h_0 = 0.09375m$).



(a) (b)

Figure 10. Profile of the normal displacement for (a) $P = 6$ and (b) $P = 10$ in example 4.3 ($N = 201$)



(a) (b)

Figure 11. Profile of the tangential displacement for (a) $P = 6$ and (b) $P = 10$ in example 4.3 ($N = 201$)

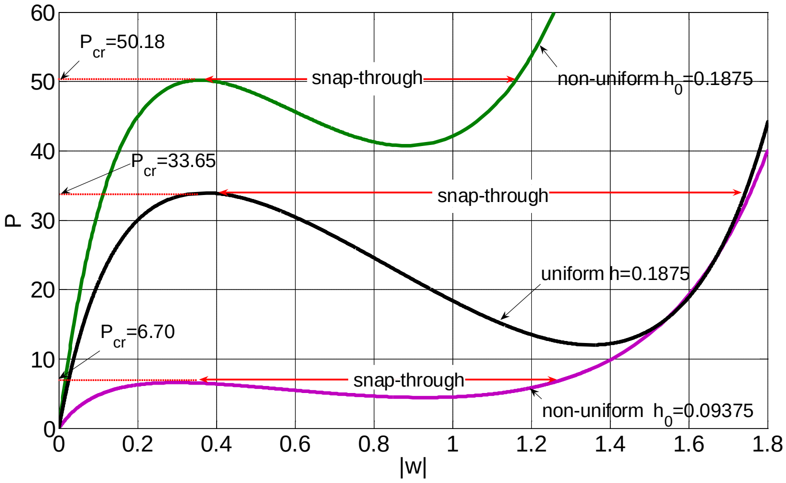


Figure 12. Load-normal displacement curves at the center of the arch in example 4.3 for uniform and non-uniform cross section ($N = 201$)

5 CONCLUSIONS

In this paper the linear and geometrically nonlinear response of non-uniform shallow arches is investigated. The solution of the derived coupled nonlinear governing equations with variable coefficients is achieved effectively using the analog equation method and results for non-uniform arches are reported for the first time in the literature. This investigation has reached to certain interesting findings concerning the employed solution method, as well as the nonlinear response of non-uniform shallow arches. These findings are summarized as follows:

1) The employed solution method exhibits stability and a small number of constant elements are adequate to obtain accurate results for the displacements and the stress resultants.
2) The displacements as well as the stress resultants are computed at any cross-section of the beam using the respective integral representations as mathematical formulae.
3) Even for small values of the external load linear analysis is inadequate to predict the real response of the arch and the use of the nonlinear one is essential.
4) The pseudo arc-length method is used for the solution of the nonlinear system of equations. This method allows following the nonlinear equilibrium path and overcome successfully bifurcation and limit points.
5) Shallow arches have increased overall stiffness contrary to straight beams, however they become unstable at a critical value of the vertical loads and a snap-through appears. Above the critical load the displacements increase suddenly to large values which may have catastrophic results in structures.
6) By changing the geometric characteristics of the arch cross-section we can alter not only the buckling load (limit point) but also the post-buckling (snap-through) response of the arch.

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