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NONLOCAL FREQUENCY ANALYSIS OF A NANOBEAM UNDER AXIAL MAGNETIC FIELD USING FINITE ELEMENT METHOD

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Abstract. *In this paper, we analyze the free transverse vibration of a nanobeam model representing the carbon nanotube that is subjected to the influence of longitudinal magnetic field. Governing equation of a nanobeam is derived employing the nonlocal elasticity theory of Eringen, Euler-Bernoulli beam theory and Maxwell classical equations. Solution for natural frequencies of a nanobeam is proposed by using the finite element method. Influences of nonlocal parameter and the magnitude of magnetic field on dimensionless natural frequencies are investigated through several numerical examples.*

1 INTRODUCTION

Recent advances in the field of nano-science [1, 2] are increasing the number of theoretical studies of nanostructures and nanocomposites [3, 4]. For prediction of vibration and stability behavior of nanostructures, many researchers successfully applied modified continuum based methods such as nonlocal theory of elasticity [5,6], which takes into account nonlocal effects appearing at the nano-scale level. Nonlocal theory is convenient to include into a model various external field effects such as magnetic or temperature field effect on the mechanical behavior of nanostructures. Murmu and Pradhan [7] investigated the thermal effects on buckling behavior of carbon nanotubes using nonlocal theory. Further, Murmu et al. [8] examined the vibration response of double-walled carbon nanotubes subjected to the influence of longitudinal magnetic field. Karličić et al. [9] investigated the dynamic behavior of the system of multiple coupled carbon nanotubes using nonlocal and viscoelasticity theory. A number of bending, vibration and buckling problems of nanostructures using nonlocal theory are solved by analytical methods. However, for more comprehensive tasks various numerical technics are used such as differential quadrature method [10], finite difference [11] or finite element method [12]. Phadikar and Pradhan [13] proposed variational formulations and finite element analysis of nanobeams and nanoplates using nonlocal elasticity models. In addition, they performed several numerical experiments to show the results for natural frequencies that will be used for the validation of the results obtained in this study.

In this work, we present a finite element formulation to examine the free transverse vibration behavior of carbon nanotube (CNT), modeled as nonlocal nanobeam, that is affected by the longitudinal magnetic field (Fig. 1). Two different cases with simply supported and clamped-clamped boundary conditions are considered. Discretization of governing equations is conducted to obtain finite element equations in terms of stiffness, mass and external force matrices. In the parametric study, effects of boundary conditions, nonlocal parameter and magnitude of longitudinal magnetic field on natural frequencies in several modes are examined in details. Further, results for natural frequencies obtained from finite element model with different number of elements are validated with the corresponding results from the literature.

2 PRELIMINARIES

2.1 Nonlocal theory

First, we consider the fundamental equations of the nonlocal elasticity and viscoelasticity theory. The key assumption in the nonlocal elasticity theory is that the stress at a point is a function of the strains at all other points of the elastic body. Based on the experimental observations, Eringen [5] derived a constitutive relation in an integral form for nonlocal stress tensor at a point **x**. The form of the nonlocal elastic relation for a three-dimensional linear, homogeneous isotropic body is given as follows

$$\sigma_{ij}(x) = \int \alpha\left(|x - x'|, \tau\right) C_{ijkl} \varepsilon_{kl}(x') dV(x'), \ \forall x \in V \tag{1}$$

$$\sigma_{ij,j} = 0 \tag{2}$$

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

where C_{ijkl} is the elastic modulus tensor for classical isotropic elasticity; σ_{ij} and ε_{kl} are the stress and the strain tensors, respectively, and u_i is the displacement vector. With $\alpha\left(|x - x'|, \tau\right)$ we denote the nonlocal modulus or attenuation function, which incorporates nonlocal effects into the constitutive equation at a reference point x produced by the local strain at a source x'. The above absolute value of the difference $|x - x'|$ denotes the Euclidean metric. The parameter $\tau = e_0 a/l$ where l is the external characteristic length (crack length, wave length), a describes the internal characteristic length (lattice parameter, granular size and distance between C-C bounds) and e_0 is a constant appropriate to each material that can be identified from atomistic simulations or by using the dispersive curve of the Born-Karman model of lattice dynamics. Since it is difficult to use the constitutive relations in integral form for solving practical problems, a simplified constitutive relation in differential form is developed. Based on works by Eringen [5], constitutive relations in differential form for the one-dimensional case is

$$\sigma_{xx} - \mu \frac{\partial \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \tag{4}$$

where *E* is elastic modulus of the beam, respectively; $\mu = (e_0 a)^2$ is the nonlocal parameter (length scales); σ_{xx} is the normal stress and $\varepsilon_{xx} = \partial u/\partial x$ is the axial deformation. Nano-size structures such as carbon nanotubes, zinc-oxid nanotubes, and other one dimensional nanostructures are modeled as nanobeams and nanorods by using the nonlocal theory, where internal characteristic lengths $e_0 a$ is often assumed to be in the range 0-2 [nm]. Putting $e_0 a = 0$ in nonlocal constitutive relation yields classical constitutive relation of the elastic body.

2.2 Maxwell relations

Based on the classical theory of electromagnetism, the relationships between the current density $\mathbf{J}$, distributing vector of magnetic field $\mathbf{h}$, strength vectors of the electric fields $\mathbf{e}$ and magnetic field permeability η are represented by Maxwell's equations in differential form and can be retrieved as

$$\mathbf{J} = \nabla \times \mathbf{h}, \ \nabla \times \mathbf{e} = -\eta \frac{\partial \mathbf{h}}{\partial t}, \ \ \nabla \cdot \mathbf{h} = 0. \tag{5}$$

Above vectors of distributing magnetic field $\mathbf{h}$ and the electric field $\mathbf{e}$ are defined as

$$\mathbf{h} = \nabla \times \left(\mathbf{U} \times \mathbf{H}\right), \ \mathbf{e} = -\eta\left(\frac{\partial \mathbf{U}}{\partial t} \times \mathbf{H}\right). \tag{6}$$

In Eq. (6), $\nabla = \partial/\partial x\,\mathbf{i} + \partial/\partial y\,\mathbf{j} + \partial/\partial z\,\mathbf{k}$ denotes the Hamilton operator, $\mathbf{U} = (x, y, z)$ is the displacement vector and $\mathbf{H} = (H_x, 0, 0)$ is the vector of the longitudinal magnetic field. We can write the vector of the distributing magnetic field in the following form

$$\mathbf{h} = -H_x\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\mathbf{i} + H_x \frac{\partial v}{\partial x}\mathbf{j} + H_x \frac{\partial w}{\partial x}\mathbf{k}, \quad \mathbf{J} = \nabla \times \mathbf{h}. \tag{7}$$

Further, using previous equations into the expressions for the Lorentz force induced by the longitudinal magnetic field, yields

$$\mathbf{f}\left(f_x, f_y, f_z\right) = \eta\left(\mathbf{J}\times\mathbf{H}\right) \tag{8}$$

$$f_x = 0,\ f_y = \eta H_x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}\right),\ f_z = \eta H_x^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial z\partial y}\right), \tag{9}$$

where f_x, f_y and f_z express the Lorentz force along the x, y and z directions.

In this study, we assume that the displacement of the *i*-th nanobeam $w_i\left(x,t\right)$ and the Lorentz force acts only in *x* direction which can be written as

$$f_{z,i} = \eta H_x^2\frac{\partial^2 w_i}{\partial x^2}. \tag{10}$$

Finally, it is possible to obtain force per unit length of the *i*-th nanobeam in the following form

$$\tilde{q}_i\left(x,t\right) = \int_A f_{z,i}dA = \eta A H_x^2\frac{\partial^2 w_i}{\partial x^2}. \tag{11}$$

3 NANOBEAM UNDER THE AXIAL MAGNETIC FIELD

3.1 Governing equation

Let us consider a simply supported homogenous nanobeam (Fig 1 b) of density ρ, cross-sectional area A and length L representing the single-walled carbon nanotube (Fig. 1 a). If nanobeam is subjected to the influence of longitudinal magnetic field, then corresponding Lorentz force resulting in force per unit length $\tilde{q}_i\left(x,t\right)$ will be induced in the axial direction of nanobeam. In addition, we consider the free transverse vibration of a nanobeam with small deflections denoted by $w(x,t)$. If we take an infinitesimal element of length dx and consider equilibrium equations for resultant forces and moments, based on Newton second and nonlocal constitutive equation (4) we can derive motion equation of the nanobeam influenced by the axial magnetic field.

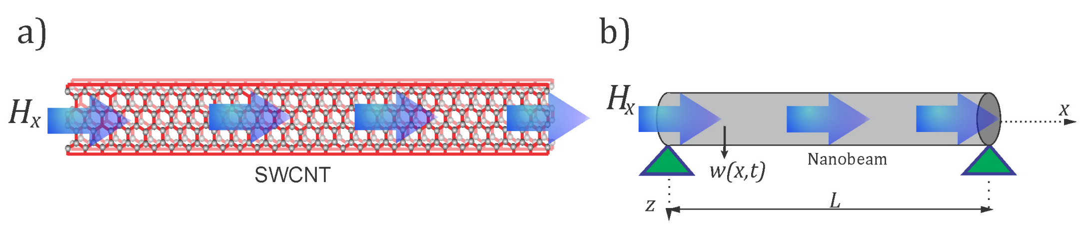

Figure 1. SWCNT as nanobeam under the influence of axial magnetic field (a) Physical model (b) Mechanical model.

First, resultant moment M and force V are written as

$$M = \mu\left[\rho A\frac{\partial^2 w}{\partial t^2} - \tilde{q}_i\left(x,t\right)\right] - EI\frac{\partial^2 w}{\partial x^2}, \tag{12}$$

$$V = \mu\left[\rho A\frac{\partial^3 w}{\partial t^2\partial x} - \frac{\partial\tilde{q}_i\left(x,t\right)}{\partial x}\right] - EI\frac{\partial^3 w}{\partial x^3}, \tag{13}$$

Then, motion equation is given in the form

$$EI\frac{\partial^4 w}{\partial x^4} + \left[1-\mu\frac{\partial^2}{\partial x^2}\right]\left[\rho A\frac{\partial^2 w}{\partial t^2} - \eta A H_x^2\frac{\partial^2 w}{\partial x^2}\right] = 0, \tag{14}$$

where μ is the nonlocal parameter, E is the Young's modulus, I denotes second moment of area, η is the magnetic field permeability and H_x denotes the magnitude of magnetic field in x direction. If we assume solution of the form $w(x,t)=U(x)e^{-i\omega t}$, we obtain the governing equation as

$$EI\frac{d^4U}{dx^4}-\left[1-\mu\frac{d^2}{dx^2}\right]\left[\rho A\omega^2 U-\eta AH_x^2\frac{d^2U}{dx^2}\right]=0\,, \tag{15}$$

3.1 Finite element formulation

In order to simplify the problem we rewrite the governing equation in the dimensionless form as

$$\frac{d^4u}{d\bar{x}^4}-\left[1-\bar{\mu}\frac{d^2}{d\bar{x}^2}\right]\left[\Omega u+MP\frac{d^2u}{d\bar{x}^2}\right]=0 \tag{16}$$

where dimensionless variables and parameters are

$$\bar{x}=\frac{x}{L}\,,u=\frac{U}{L}\,,\ \bar{\mu}=\frac{\mu}{L}\,,\ \Omega=\sqrt{\frac{\omega^2\rho AL^4}{EI}}\,,\ MP=\frac{\eta AH_x^2L^2}{EI}\,. \tag{17}$$

Weak form of Eq. (16) is given as

$$\begin{aligned}0&=\int_{x_e}^{x_{e+1}}\left[\frac{d^4u}{d\bar{x}^4}-\left[1-\bar{\mu}\frac{d^2}{d\bar{x}^2}\right]\left[\Omega u+MP\frac{d^2u}{d\bar{x}^2}\right]\right]vdx\\&=\int_{x_e}^{x_{e+1}}\left[v\frac{d^4u}{d\bar{x}^4}-\Omega vu+MP\,v\frac{d^2u}{d\bar{x}^2}+\bar{\mu}\,\Omega v\frac{d^2u}{d\bar{x}^2}+\bar{\mu}MP\,v\frac{d^4u}{d\bar{x}^4}\right]dx\\&=\int_{x_e}^{x_{e+1}}\left[\left(1+\bar{\mu}MP\right)\frac{d^2v}{d\bar{x}^2}\frac{d^2u}{d\bar{x}^2}+\left(MP-\bar{\mu}\,\Omega\right)\frac{dv}{d\bar{x}}\frac{du}{d\bar{x}}+\Omega vu\right]dx\\&\quad-v\left[\bar{V}-\frac{du}{d\bar{x}}\right]_{x_e}^{x_{e+1}}-\frac{dv}{d\bar{x}}\left[\bar{M}-\bar{\mu}\,\Omega u\right]_{x_e}^{x_{e+1}}.\end{aligned} \tag{18}$$

The finite element model is derived by using the Hermite interpolation functions for the element with two nodal degrees of freedom $\left(u,-\frac{du}{dx}\right)$. For the finite element, the interpolation of the displacement in local coordinate system is written as

$$\bar{u}^e=\sum_{j=1}^{4}\Delta_j^e\phi_j^e\ , \tag{19}$$

where Δ_j^e are nodal variables and ϕ_j^e are Hermite interpolation functions. Taking into account Eq. (19) in Eq. (18) and taking that $v=\phi_i^e$, we obtain the *i*-th algebraic equation in the form

$$0=\sum_{j=1}^{4}\left[\int_{x_e}^{x_{e+1}}\left(\left(1+\bar{\mu}MP\right)\frac{d^2v}{d\bar{x}^2}\frac{d^2u}{d\bar{x}^2}+\left(MP-\bar{\mu}\,\Omega\right)\frac{dv}{d\bar{x}}\frac{du}{d\bar{x}}+\Omega vu\right)dx\right]\Delta_j^e-Q_i^e\,. \tag{20}$$

Then, the finite element model is given as

$$\left(\left[K^e\right]-\Omega\left[M^e\right]\right)=\left\{Q^e\right\} \tag{21}$$

where

$$K_{ij}^{e}=\int_{x_e}^{x_{e+1}}\left(\left(1+\bar{\mu}MP\right)\frac{d^2\phi_i^e}{d\bar{x}^2}\frac{d^2\phi_j^e}{d\bar{x}^2}+MP\frac{d\phi_i^e}{d\bar{x}}\frac{d\phi_j^e}{d\bar{x}}+\right)dx\,, \tag{22}$$

$$M_{ij}^{e}=\int_{x_e}^{x_{e+1}}\left(\phi_i^e\phi_j^e+\bar{\mu}\frac{d\phi_i^e}{d\bar{x}}\frac{d\phi_j^e}{d\bar{x}}\right)dx\,, \tag{23}$$

$$Q_1^e=-\left[\bar{V}-\frac{du}{dx}\right]_{x_e},\; Q_2^e=-\left[\bar{M}-\bar{\mu}\Omega u\right]_{x_e},\; Q_3^e=\left[\bar{V}-\frac{du}{dx}\right]_{x_{e+1}},\; Q_4^e=\left[\bar{M}-\bar{\mu}\Omega u\right]_{x_{e+1}}. \tag{24}$$

4 NUMERICAL RESULTS AND DISCCUSIONS

We developed corresponding numerical codes to compute the solution for dimensionless frequencies of the nanobeam subjected to the influence of longitudinal magnetic field. First, we validate our results with the results for dimensionless natural frequency obtained in [12]. Further, in the numerical analysis we have observed the effects of magnetic field and nonlocal parameter on dimensionless frequency. Two different types of boundary conditions on nanobeam are considered, simply supported and cantilever beam. The values of parameters are given in figures. In all simulations, we used 24 elements, which yield satisfying solutions (see Table 1). However, the minimum i.e. sufficient number of elements need to obtain the convergence solution is not examined here.

In Table 1, we show the numerical results for dimensionless natural frequencies obtained by using the finite element model of nonlocal Euler-Bernoulli beam in reference [12] and our results obtained from Eq. (22) when the magnetic field parameter $MP=0$ is neglected. It can be observed that our results for natural frequencies are in good agreement with the results obtained in [12].

Nonlocal parameter	Natural frequency Ref. [12]	Natural frequency Present study
0	9.8697	9.8696
	39.4848	39.4785
	88.8984	88.8279
1	2.9936	2.9936
	6.2061	6.2051
	9.3796	9.3723

Table 1: Dimensionless natural frequency for the simply supported case.

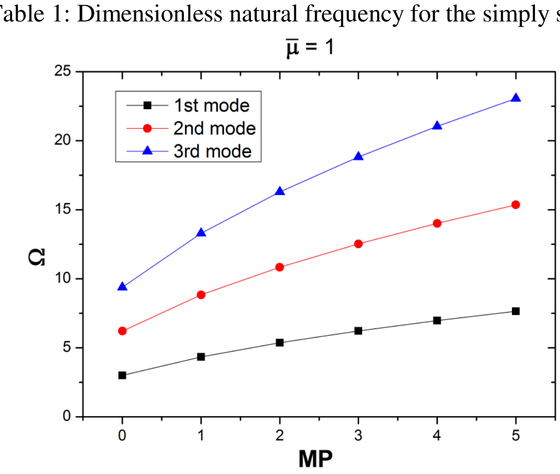

Figure 2. Dimensionless natural frequency for changes of magnetic field parameter and simply supported nanobeam.

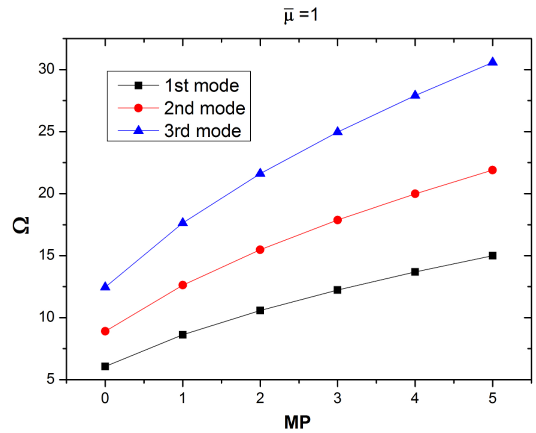

Figure 3. Dimensionless natural frequency for changes of nonlocal parameter and clamped-clamped nanobeam.

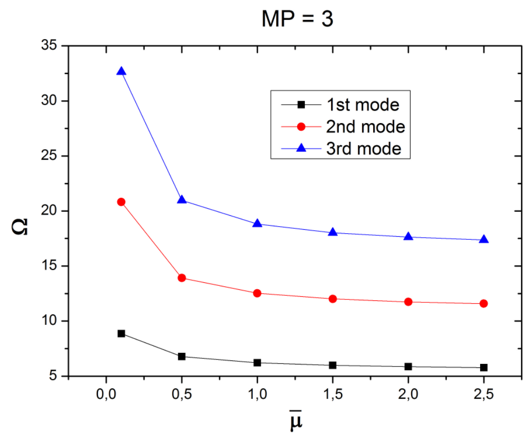

Figure 4. Dimensionless natural frequency for changes of nonlocal parameter and simply supported nanobeam.

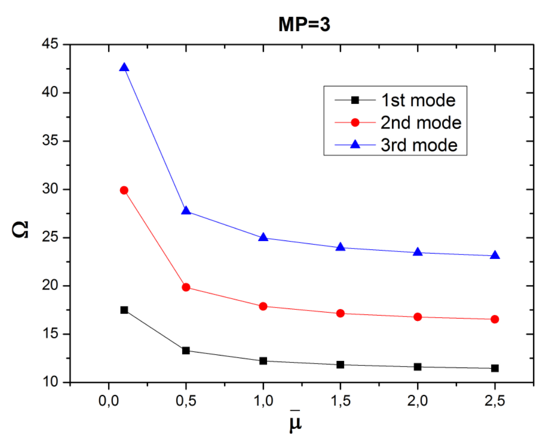

Figure 5. Dimensionless natural frequency for changes of nonlocal parameter and clamped-clamped nanobeam.

Further, we investigate the influnce of magnetic field parameter on the dimensionless frerquency for fixed value of nonlocal parameter $\bar{\mu}=1$. From Figs. 1 and 2, we can see that an increase of magnetic field parameter increases the value of frequency in all three vibration modes and both boundary conditon cases, simply supported and clamped-clamped nanobeam. However, this increase is more pronounced in higher vibration modes than in the lower ones. Such effect of magnetic filed is attributed to the increase of overall stifnees of the the system by increasing the magnitude of magnetic field. Also, it can be noticed that the frequencies are highre in the clamped-clamped case due to the the stronger constraints.

Figs. 4 and 5 shows the influence of nonlocal parameter on dimensionless frequency for fixed value of magnetic field parameter in first three vibration modes and two types of boundary conditions**.** It can be noticed that the the frequency decreases for an increase of the nonlocal parameter. This effect of decrease is more pronounced for lower values of nonlocal parameter but it is weakening for higher values of the parameter. Here, we also have higher values of natural frequencies for the clamped-clamped nanobeam as expected. The obtained results are in line with the corresponding results in the literature obtained by other authores.

5 CONCLUSION

In this paper, the finite element approach is used to examine the vibration behavior of nanobeam modeled via nonlocal elasticity theory and subjected to the influence of axial magnetic field. The effects of the magnitude of magnetic field, nonlocal parameter and boundary conditions on dimensionless natural frequencies are examined in the first three vibration modes through several numerical examples. The results obtained by the finite element method are in line with other results found in the literature. The exerted magnetic field increases the natural frequency of the system due to the increase in overall stiffness of the system. An increase of nonlocal parameter, which represents the nonlocal effects on nano-scale level, leads to a decrease of frequency compared to the classical local elasticity models. This study can be applied in future vibration and stability finite element analysis of more complex nanostructure and nanocomposite systems with various external field effects included. In addition, it can be useful in design procedures of modern nano-devices like nanosensors and nanoresonators.

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