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CONTROLLED LAGRANGIANS AND ROBUST STABILIZATION OF A NONLINEAR UNDERACTUATED MICROBEAM WITH ROTATING JOINT

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Abstract. *In this paper the rigid-flexible coupling dynamical model and the nonlinear governing equations of motion of a microbeam attached to a rotating hub as an application of under-actuated systems will be developed. The coupling model is derived employing the strain gradient elasticity theory, taking the geometrically nonlinearities of the microbeam into account. The Rayleigh–Ritz method is used to discretize partial differential equation to obtain a set of nonlinear ordinary differential equations of motion. Then, controlled Lagrangian method as a robust procedure for controller design is employed to achieve an acceptable regulation on the hinge's angle of rotation of micro-cantilever beam while undesirable vibration of the under-actuated flexible variable is damped. Since the controlled system is Lagrangian by construction, energy methods can be utilized to find control gains that yield closed-loop stability. The advantages of controlled Lagrangian method lie in inference of the stabilization in terms of energy and its capability of handling under-actuated devices. The effect of controller gains on stability and robustness of the controller in presence of structured uncertainties is investigated analytically. Then, the performance of designed control scheme is illustrated through several numerical simulation results and some comparisons are made in various situations.*

1 INTRODUCTION

The extensive applications of microbeams motivated considerable amount of research on this topic. Micro-scale structures are widely applied in many modern machine components, such as micro probes, micro actuators, micro manipulators, biosensors, micro switches, electrostatically excited micro actuators, and vibration shock sensors [1-5].

Although the literature regarding the linear and nonlinear statics and dynamics of microbeams are quite large [6,7], however, little effort has been made to control the vibration in microbeams associated with non-classical theories such as strain gradient theory. Authors in [8] investigate the problem of vibration control of strain gradient micro-cantilevers by using the Partial Differential Equation (PDE) control theory. A sliding mode controller is proposed for a micro-cantilever beam with fringing and squeezed film damping effects in [9].

The system of a micro-cantilever beam with rotating joint is considered as an underactuated system because of the uncontrolled modes of microbeam which constitute the internal dynamics of the system. A recent approach to the control of underactuated systems is to look for control laws which will induce some specified structure on the closed loop system. The method of controlled Lagrangian (CL) is a control design approach based on energy method for underactuated Lagrangian systems [10]. Underactuated systems are often not even linearly controllable. Even for systems which are linearly controllable, using nonlinear control techniques is often desirable to retain the basically nonlinear dynamics and can provide stability results that hold over larger domain than can be obtained using linear design and analysis. This method was first proposed in the paper by Bloch, Krishnaprasad, Marsden and Sánchez [11], where it was shown that an internal rotor can be used to effectively shape the kinetic energy of a spacecraft in order to stabilize steady rotation about the intermediate principal axis of inertia.

Considerable efforts have then been done to control some underactuated and flexible applications by this method [12,13].

In this paper, the controlled Lagrangian methodology as an energy shaping control design method which could be appealing as it retains the underlying nonlinear dynamics is presented for a rotation-hinge nonlinear microbeam. This control methodology is uniquely formulated without using any form of feed-forward compensation for the system uncertainties or nonlinearities. A stability analysis is also conducted in which the closed-loop dynamics is theoretically proven to achieve the specified generalized coordinates and thereby, information on how to choose the control gains to achieve closed-loop stability is obtained. In addition, taking into account the parameter uncertainties, stability robustness of the system is analytically investigated.

2 DYNAMICAL MODEL AND GOVERNING EQUATIONS

In this section dynamics of a micro-cantilever beam with rotating joint is developed. The system is considered as shown in Figure 1. This beam is assumed as an Euler–Bernoulli beam, so the shear deformation and rotary inertia are neglected and the joint is assumed to be rigid. The type of nonlinearity is geometric, due to the stretching effect of the mid-plane of the microbeam. The microbeam cross-section is constant over the entire length of the beam and only the transverse displacement is considered.

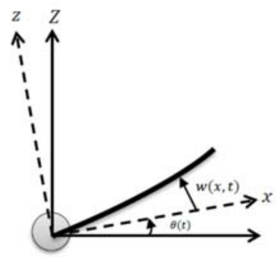


Figure 1- Schematic of a single-link micromanipulator

Considering a geometrically nonlinear axial strain–displacement equation and the strain gradient elasticity theory, the potential energy of the microbeam can be obtained as [14];

$$U = \frac{1}{2}EA\left(\int_0^l\left(\frac{1}{4}\left(\frac{\partial w(x,t)}{\partial x}\right)^4 + \frac{1}{2}l^2\left(\left(\frac{\partial w(x,t)}{\partial x}\right)^2\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 + \left(\frac{\partial w(x,t)}{\partial x}\right)^3\left(\frac{\partial^3 w(x,t)}{\partial x^3}\right)\right)\right)dx\right) + \frac{1}{2}EI\left(\int_0^l\left(\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 + l^2\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)\left(\frac{\partial^4 w(x,t)}{\partial x^4}\right)\right)dx\right). \tag{1}$$

in which w(x) is z-component of the displacement vector, , E includes the elastic moduli, and l denotes the internal length scale. Deriving the velocity of an element with infinitesimal mass dm on the microbeam and neglecting the higher order of transverse displacement due to its small values, the kinetic energy of the rotary microbeam is then obtained as follows [15]:

$$T = \frac{1}{2}\rho A\left(\int_0^l\left(\left(\frac{\partial w(x,t)}{\partial t}\right)^2 + 2x\left(\frac{\partial w(x,t)}{\partial t}\right)\left(\frac{d\theta(t)}{dt}\right) + x^2\left(\frac{d\theta(t)}{dt}\right)^2\right)dx\right) + \frac{1}{2}J_h\left(\frac{d\theta(t)}{dt}\right)^2 \tag{2}$$

where J_h is hub moment of inertia.

Rayleigh–Ritz method is employed to expand $w(x,t)$ in a finite series to obtain a set of nonlinear ordinary differential equations governing the motion of the system from Eqs. (1) and (2), using the [16],

$$w(x,t) = \sum_{r=1}^{N}\varphi_r(x)q_r(t) \qquad (r = 1,2,\dots,N) \tag{3}$$

in which $\varphi_r(x)$ represents the rth assumed mode shape functions for the transverse motion of a clamped-free beam and $q_r(t)$ denotes the rth time dependent, displacement generalized coordinates. Substituting (3) into (1) and (2) and then utilizing Euler-Lagrangian formulism [17], the governing equations of the micro-cantilever beam with rotating joint can be given

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_f(t) \\ \ddot{q}_\theta(t) \end{Bmatrix} + \begin{bmatrix} K_1 q_f(t) - K_2 q_f{}^3(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \tag{4}$$

where $q_f(t)$ is the microbeam deflection coordinate, $q_\theta(t)$ is hub's angle of rotation, $\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$ represents the total inertia matrix, K_1 and K_2 are the linear and cubic stiffness coefficients of the system, respectively; the cubic stiffness coefficient is responsible for the hardening behavior of the system [18]., and τ includes the applied external torques.

3 CONTROLLED LAGRANGIAN

The purpose of this section is to present a controlled Lagrangian design procedure for a rotary micro-cantilever beam. Here the system is considered as a 2-DOF underactuated mechanical system with hub's rotation and arm's deflection as the two coordinates in the presence of moment input. Defining errors as $\tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q_d}$, where q_d is the desired trajectory for the robot such that $q_{fd} = 0$ and $\ddot{q}_{\theta d} = 0$, equation (4) can be rewritten as follow:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{q}}_f(t) \\ \ddot{\tilde{q}}_\theta(t) \end{Bmatrix} + \begin{bmatrix} K_1(\tilde{q}_f + q_{fd}) - K_2(\tilde{q}_f + q_{fd})^3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau = \boldsymbol{W}\tau \tag{5}$$

It is worthwhile noting that the control bundle $\boldsymbol{W} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $rank(W) = 1$, guaranties that the system is underactuated.

The point of interest is $\tilde{q} = (0,0)$, which corresponds to zero tracking error in the rigid variable and null deflections. Controlled Lagrangian method as explained in [19] is a control strategy for finding a Lagrangian with center equilibrium point by shaping kinetic and potential energies through solving matching equation presented as the follows and injecting dissipative force to obtain stable-focus equilibrium point.

$$\boldsymbol{W} = [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}\widehat{\boldsymbol{W}} \tag{6}$$

$$[\boldsymbol{W}^\perp]\left[[\boldsymbol{M}]\{\ddot{\boldsymbol{q}}\} + [\boldsymbol{C}]\{\dot{\boldsymbol{q}}\} + \{\boldsymbol{g}(\boldsymbol{q})\} - \boldsymbol{F} - [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}([\widehat{\boldsymbol{M}}]\{\ddot{\boldsymbol{q}}\} + [\widehat{\boldsymbol{C}}]\{\dot{\boldsymbol{q}}\} + \{\widehat{\boldsymbol{g}}(\boldsymbol{q})\} - \widehat{\boldsymbol{F}})\right] = \boldsymbol{0} \tag{7}$$

Equations (6) and (7) depict the first and the second matching equations, respectively. In which W is control bundle, $W^\perp$ is the Left annihilator of W space ($[W^\perp][W] = 0$), $[M]$, $[C]$, $\{g(\boldsymbol{q})\} = \frac{\partial U(\boldsymbol{q})}{\partial q^k}$, and F are mass matrix of inertia, Christoffel symbol of first kind, nonlinear stiffness matrix resulting from potential energy $U(\boldsymbol{q})$, and external forces vector, respectively. Also, symbols with hat sign have the same rules and belong to the second Lagrangian system which is tried to find. Considering the second matching equation and collecting velocity dependent terms, one can divide this equation in two sub equations which are presented as follow:

$$[\boldsymbol{W}^\perp]\left[\left([\boldsymbol{C}] - [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}[\widehat{\boldsymbol{C}}]\right)\{\dot{q}\} - \boldsymbol{F}^v(q,\dot{q}) + [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}\widehat{\boldsymbol{F}}^v(q,\dot{q})\right] = \boldsymbol{0} \tag{8}$$

$$[\boldsymbol{W}^\perp]\left[\{\boldsymbol{g}\} - [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}[\widehat{\boldsymbol{g}}] - \boldsymbol{F}^q(q) + [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}\widehat{\boldsymbol{F}}^q(q)\right] = \boldsymbol{0} \tag{9}$$

Where $\boldsymbol{F^q}$ and $\boldsymbol{F^v}$ are velocity independent and velocity dependent parts of external force F, i.e. the gyroscopic and the dissipative forces. Also, $\boldsymbol{q} = [q_f, q_\theta]$ and $\dot{\boldsymbol{q}} = [\dot{q}_f, \dot{q}_\theta]$ represent generalized coordinates and velocities, respectively. It should be noted that for a constant inertia tensor M, which is true for our case of a rotating microbeam, the Christoffel coefficients of the first kind will be equal to zero (i.e. [C]=0).

First, we deal with the items regarding the controlled kinetic energy. Considering the governing equations of motion, (5), and the matching equation, (8), assuming that there is no need to consider gyroscopic terms for the second Lagrangian systemi, for free vibration of the system (no external forces, i.e. $F^v(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$), the simplest form of solution of the partial differential equation (8) is considering constant symmetric matrix for $\widehat{\mathrm{M}}^{-1}$ as considered below:

$$\widehat{\boldsymbol{M}}^{-1} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \tag{10}$$

[i] This assumption is usually made for simplicity of the procedure; if it is not held true, we need to repeat the procedure considering the gyroscopic terms for the second Lagrangian system. In our case of a rotating microbeam, fortunately this assumption is correct and there is no need to repeat the process.

Where a_1, a_2, and a_3 are arbitrary constants. Assuming matrix $\widehat{M}$, general solution of the potential shaping equation (equation. (9)) is then obtained as follow:

$$\widehat{U}(q_1, q_2) = -\frac{1}{4}\left(\frac{K_2 {q_f}^4}{m_{11}a_1 + m_{12}a_2}\right) + \frac{1}{2}\left(\frac{K_1 {q_f}^2}{m_{11}a_1 + m_{12}a_2}\right) + F\left(\frac{-q_f m_{11}a_2 - q_f m_{12}a_3 + q_\theta m_{11}a_1 + q_\theta m_{12}a_2}{m_{11}a_1 + m_{12}a_2}\right) \tag{11}$$

Where $\widehat{U}$ is the potential energy function of the second Lagrangian system, and $F(.)$ is an arbitrary differentiable function with respect to its own arguments.

In order to obtain control bundle of matching system, first matching equation (equation (6)) is solved resulting in:

$$\widehat{\boldsymbol{W}} = [\widehat{\boldsymbol{M}}][\boldsymbol{M}]^{-1}[\boldsymbol{W}] = \begin{bmatrix} -\frac{m_{12}a_3}{(a_1a_3 - {a_2}^2)(m_{11}m_{22} - {m_{12}}^2)} - \frac{m_{11}a_2}{(a_1a_3 - {a_2}^2)(m_{11}m_{22} - {m_{12}}^2)} \\ \frac{m_{12}a_2}{(a_1a_3 - {a_2}^2)(m_{11}m_{22} - {m_{12}}^2)} + \frac{m_{11}a_1}{(a_1a_3 - {a_2}^2)(m_{11}m_{22} - {m_{12}}^2)} \end{bmatrix} = \begin{bmatrix} P_{q_f} \\ P_{q_\theta} \end{bmatrix} \tag{12}$$

The energy shaping control u_{es} is given by:

$$u_{es} = \{\boldsymbol{g}(q)\} - [\boldsymbol{M}][\widehat{\boldsymbol{M}}]^{-1}\{\widehat{\boldsymbol{g}}(q)\} \tag{13}$$

The controller design is completed with a second term, corresponding to the damping injection. We deal with the situation in the presence of Rayleigh dissipative forces. Choosing dissipative forces:

$$u_{diss} = -\omega\left(P_{q_f}.\dot{q}_f + P_{q_\theta}.\dot{q}_\theta\right), \quad \omega > 0 \tag{14}$$

Eventually, the stabilizing control law for the conservative system as presented in [20] is

$$u = u_{es} + [\boldsymbol{W}]u_{diss} \tag{15}$$

Where the first term is designed to achieve the energy shaping and the second one injects the damping.

4 STABILITY ANALYSIS

According to equation (14), It can be proved this force is dissipative since:

$$\frac{d\hat{E}}{dt} = \langle\dot{\boldsymbol{q}}; \hat{u}\rangle = P_{q_f}.\dot{q}_f.u_{diss} + P_{q_\theta}.\dot{q}_\theta.u_{diss} = -\omega\left(P_{q_f}.\dot{q}_f + P_{q_\theta}.\dot{q}_\theta\right)^2 \leq 0. \tag{16}$$

Therefore, the damping constant ω has a direct relationship with decreasing the energy of close-loop system. To analyze the stability of the closed-loop system we can consider the controlled energy of system as its Lyapunov function candidate:

$$\hat{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \hat{E} = \hat{T} + \widehat{U} \tag{17}$$

Where kinetic energy and potential energy of matching system are obtained through matching conditions described above and are defined as respectively,

$$\hat{T} = \frac{1}{2}\frac{a_3{\dot{q}_f}^2 - 2a_2\dot{q}_f\dot{q}_\theta + a_1{\dot{q}_\theta}^2}{a_1a_3 - {a_2}^2} \tag{18}$$

$$\widehat{U}(q_1, q_2) = -\frac{1}{4}\left(\frac{K_2 {q_f}^4}{m_{11}a_1 + m_{12}a_2}\right) + \frac{1}{2}\left(\frac{K_1 {q_f}^2}{m_{11}a_1 + m_{12}a_2}\right) + \frac{1}{2}\varepsilon\left(\frac{-q_f m_{11}a_2 - q_f m_{12}a_3 + q_\theta m_{11}a_1 + q_\theta m_{12}a_2}{m_{11}a_1 + m_{12}a_2}\right)^2 \quad (19)$$

It should be noted that the arbitrary differentiable function, $F(.)$ in the general solution of $\widehat{U}(q_1, q_2)$ as mentioned in the following equation is defined as the following:

$$F = \frac{1}{2}\varepsilon\left(\frac{-q_f m_{11}a_2 - q_f m_{12}a_3 + q_\theta m_{11}a_1 + q_\theta m_{12}a_2}{m_{11}a_1 + m_{12}a_2}\right)^2 \quad (20)$$

In order to achieve stability, the positive condition should be imposed on controlled energy of system. Hessian matrix of $\hat{E}(q, \dot{q})$ should be positive definite to hold positive definite conditions of controlled energy. The Hessian Matrix is defined as follow:

$$\delta^2 \hat{E} = \begin{bmatrix} h_1 & -\frac{\varepsilon(a_2 m_{11}+a_3 m_{12})}{a_1 m_{11}+a_2 m_{12}} & 0 & 0 \\ -\frac{\varepsilon(a_2 m_{11}+a_3 m_{12})}{a_1 m_{11}+a_2 m_{12}} & \varepsilon & 0 & 0 \\ 0 & 0 & \frac{a_3}{(a_1 a_3 - {a_2}^2)} & -\frac{a_2}{(a_1 a_3 - {a_2}^2)} \\ 0 & 0 & -\frac{a_2}{(a_1 a_3 - {a_2}^2)} & \frac{a_1}{(a_1 a_3 - {a_2}^2)} \end{bmatrix}, \quad (21)$$
$$h_1 = \frac{-3K_2 q_f(t)^2}{a_1 M_{ff} + a_2 M_{f\theta}} + \frac{K_1}{a_1 M_{ff} + a_2 M_{f\theta}} + \frac{\varepsilon\left(a_2 M_{ff} + a_3 M_{f\theta}\right)^2}{\left(a_1 M_{ff} + a_2 M_{f\theta}\right)^2}.$$

Therefore, the controlled energy remains positive near its minimum equilibrium point and can be used as the Lyapunov candidate. Then, the time derivative of controlled energy is semi-negative for $\omega > 0$ as shown in equation (16), and this will guarantee the stability of the closed-loop system from LaSalle's lemma. In order to obtain all the solutions of the closed-loop system that asymptotically converge to largest invariant set, these following inequalities have to be satisfied which are helpful to determine controller parameter range. The $-3K_2 {q_f}^2$ term when comparing with K1 is neglected due to small amount of q_f.

$$\frac{\varepsilon K_1}{(a_1 m_{11} + a_2 m_{12})(a_1 a_3 - {a_2}^2)} > 0 \quad (22)$$
$$\frac{\varepsilon K_1 a_3}{(a_1 m_{11} + a_2 m_{12})(a_1 a_3 - {a_2}^2)} > 0 \quad (23)$$
$$\frac{K_1}{a_1 m_{11} + a_2 m_{12}} + \frac{\varepsilon(a_2 m_{11} + a_3 m_{12})^2}{(a_1 m_{11} + a_2 m_{12})^2} > 0 \quad (24)$$
$$\frac{\varepsilon K_1}{a_1 m_{11} + a_2 m_{12}} > \quad (25)$$

which yields the range of controller parameters as follow:

$$\varepsilon > 0, \quad a_3 > 0, \quad a_1 a_3 - {a_2}^2 > 0, \quad a_1 m_{11} + a_2 m_{12} > 0 \quad (26)$$

To investigate the performance and the stability of the designed controller in the presence of the structural uncertainties, which are responsible for the difference among the actual and the determined values of the system's parameters, the structural uncertainties should be considered in the equation of motion of the system. The equation (4) is represented in a compact form as follow:

$$[\boldsymbol{M}]\begin{Bmatrix} \ddot{q}_i(t) \\ \ddot{\theta}(t) \end{Bmatrix} + \begin{Bmatrix} g_f(q_i(t)) \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \quad (27)$$

where $[\boldsymbol{M}]$ represents the total inertia matrix, and g_f is component of the nonlinear stiffness depends on the elasticity of the microbeam. The structural uncertainties of the system are modeled in a general form of nominal plus perturbation model as the following:

$$\boldsymbol{M} = \bar{\boldsymbol{M}} + \tilde{\boldsymbol{M}}$$
$$g_f = \bar{g}_f + \tilde{g}_f \tag{28}$$

In order to map the uncertain terms from the open-loop system on the second Lagrangian system, these terms should be modeled as an external force as the following:

$$F = -\tilde{\boldsymbol{M}} \begin{Bmatrix} \ddot{q}_f(t) \\ \ddot{q}_\theta(t) \end{Bmatrix} - \tilde{\boldsymbol{g}} \tag{29}$$

Then, these terms may be mapped through matching condition (8) which is represented as:

$$\hat{F} = -\widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{M}} \begin{Bmatrix} \ddot{q}_f(t) \\ \ddot{q}_\theta(t) \end{Bmatrix} - \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{g}} \tag{30}$$

Exerting this force in the equation of motion of the second Lagrangian system $\hat{L}$, the governing equations of motion modify as the following:

$$\widehat{\boldsymbol{M}} \begin{Bmatrix} \ddot{q}_f(t) \\ \ddot{q}_\theta(t) \end{Bmatrix} + \widehat{\boldsymbol{g}} = \widehat{\boldsymbol{u}} - \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{M}} \begin{Bmatrix} \ddot{q}_f(t) \\ \ddot{q}_\theta(t) \end{Bmatrix} - \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{g}} \tag{31}$$

$$\left(\widehat{\boldsymbol{M}} + \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{M}}\right) \begin{Bmatrix} \ddot{q}_f(t) \\ \ddot{q}_\theta(t) \end{Bmatrix} + \left(\widehat{\boldsymbol{g}} + \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{g}}\right) = \hat{u} \tag{32}$$

One can conclude that (32) is the equation of motion of a system with the mass matrix of inertia $\widehat{\mathrm{M}} + \widehat{\mathrm{M}}\mathrm{M}^{-1}\tilde{\mathrm{M}}$, and the potential energy $\widehat{U} + \widetilde{U}$ which is governing in the following relation:

$$\widehat{\boldsymbol{g}} + \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{g}} = \frac{\partial \widehat{U}(q)}{\partial q^k} + \frac{\partial \widetilde{U}(q)}{\partial q^k} \tag{33}$$

Now, the stability of the system should be examined using Hessian Matrix of the controlled energy:

$$\delta^2 E_{2n\times 2n} = \begin{bmatrix} \left[\dfrac{\partial^2\left(\widehat{U} + \widetilde{U}\right)}{\partial q^i \partial q^j}\right] & 0 \\ 0 & \left[\widehat{\boldsymbol{M}} + \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{M}}\right] \end{bmatrix} \tag{34}$$

It should be noted that in the equation (34), the second derivative of the potential function is required and it can be obtained determining the derivative of the term $\widehat{\boldsymbol{g}} + \widehat{\boldsymbol{M}}\boldsymbol{M}^{-1}\tilde{\boldsymbol{g}}$. Therefore, there is no need to extract the explicit form of that function. equation (34) leads one to conclude that if the constants a_1, a_2 and a_3 would be selected with an appropriate margin from their limit values which are expressed in equation (26), the designed controller represents stability robustness in the presence of the changes of structural parameters of the system. In addition, the allowable deviation of the uncertain parameters $\widetilde{M}$ and $\tilde{g}$ can be determined by assuming the positive condition of the Hessian Matrix in equation (34) where all constants are assigned to their values. It is worthwhile to note that two constants ω and ε do not contribute in stability robustness of the closed-loop system and only provide the proper performance.

5 SIMULATION RESULTS

In this section, by using some comparisons and conducting several numerical simulations, the controller design for different situations is verified. The above described control scheme is now used to control the states of a rotary micro-cantilever beam. The geometry and material properties of the system are listed in Table 1. Figure 2 shows the response for the zero reference input in absence of injected damping force with initial conditions of $\mathrm{q}_f = 10^{-7}\mu m$, $\mathrm{q}_\theta = 0$, and $\dot{\boldsymbol{q}} = 0$ for regulation control goal. As it can be observed in Figure 2(c), total energy of closed-loop system remains constant cause to center equilibrium point which obtained through matching conditions as discussed before. Phase diagram of flexible variable can be observed in Figure 2(d) which is compatible with the prior explanation. Here the control goal is hinge's angle of rotation regulation and vibration suppression of flexible deflection by injecting damping force. The appropriate values of a_1, a_2, and a_3 have been used to satisfy the conditions (26). As it can be seen from Figure 3 beam vibration and total energy of close-loop system decay toward zero as time increases while joint's angle is regulated. Comparing Figure 2(d) and 2(d) which indicate plane phase diagram of flexible deflection, it can be illustrated that the system converges to its stable equilibrium point asymptotically after injecting damping force. Considering Figure 3(b), it can be obviously observed that the equilibrium point became stable-focus after damping injection and the total energy of close-loop system decays toward zero as system reaches to its equilibrium point.

Constant ε determines the weight of potential function of the system which has effect in underactuated coordinate. Generally by increasing ε and a_i the total energy of second lagrangian increases which cause to increasing the flexible deflection oscillation. All constants a_i, ω and ε should be adjusted simultaneously using trial and error procedure to avoid system oscillation.

In Figure 4 the numerical calculations are carried out to validate the analytical results obtained formerly regarding the robustness stability of the closed-loop system in taking into account the parametric uncertainties. In particular, the uncertainty in length parameter of the microbeam is considered, i.e. $l = \bar{l} + \tilde{l}$, which has contribution in inertia matrix, $\tilde{\boldsymbol{M}}$ and the stiffness vector, $\tilde{\mathrm{g}}_{\mathrm{f}}$. In simulations, all controller gains assumed fix and the response curves of the system is plotted during altering the uncertain parameter, $\tilde{l}$. As it is illustrated in Figure 4(a) to Figure 4(f), the studied system remains stable for the values beyond $\tilde{l} = -0.07$ and will be unstable for values of $\tilde{l}$ less than -0.08.

Material	Length (μm)	Width (μm)	Thickness (μm)	Young's modulus (GPa)	Density ($\frac{kg}{m^3}$)	Joint moment of inertia $Kg.\mu m^2$
S_iO_2	500	30	3	107	2330	2.6×10^{-5}

Table 1- Physical properties of the rotary microbeam.

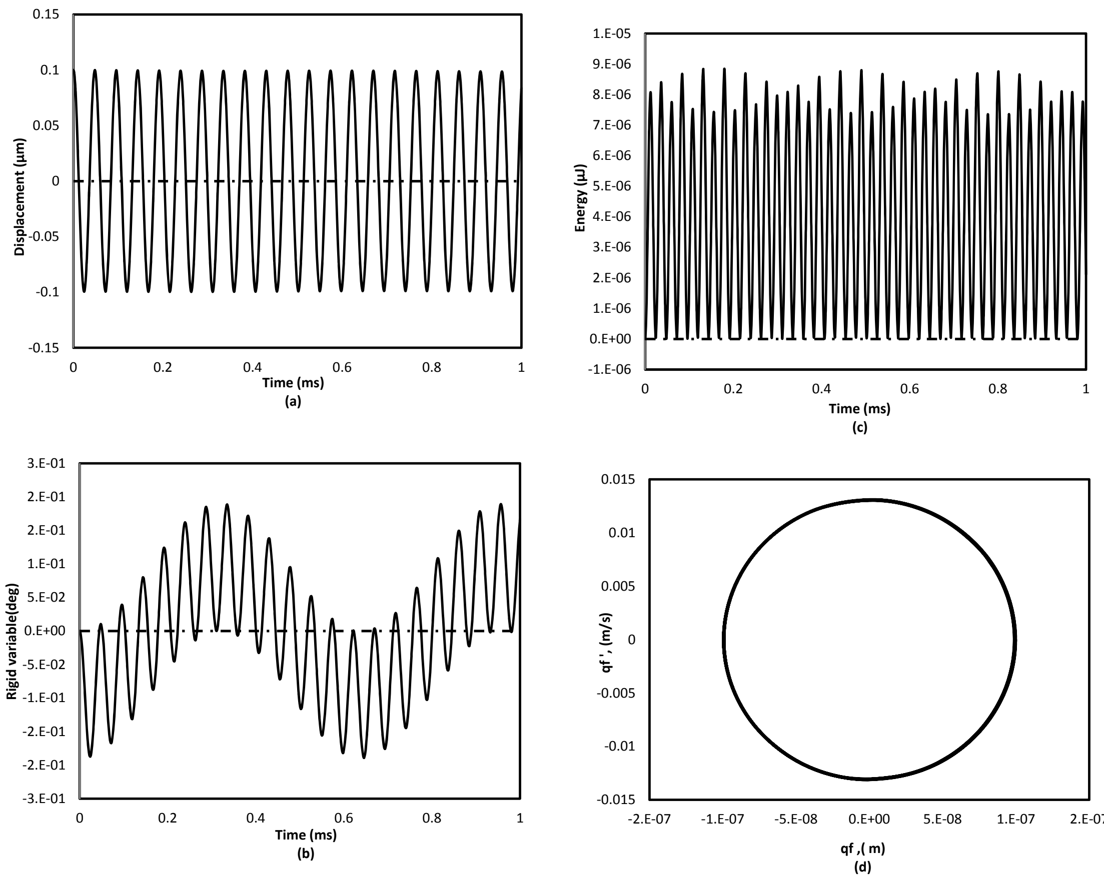


Figure 2. Simulation results for CL control in absence of injected damping force: (a) Time evolution of the beam displacement; (b) Time evolution of the hub's angle of rotation (rigid variable); (c) Total energy of closed-loop system; (d) plane phase diagram of flexible deflection.

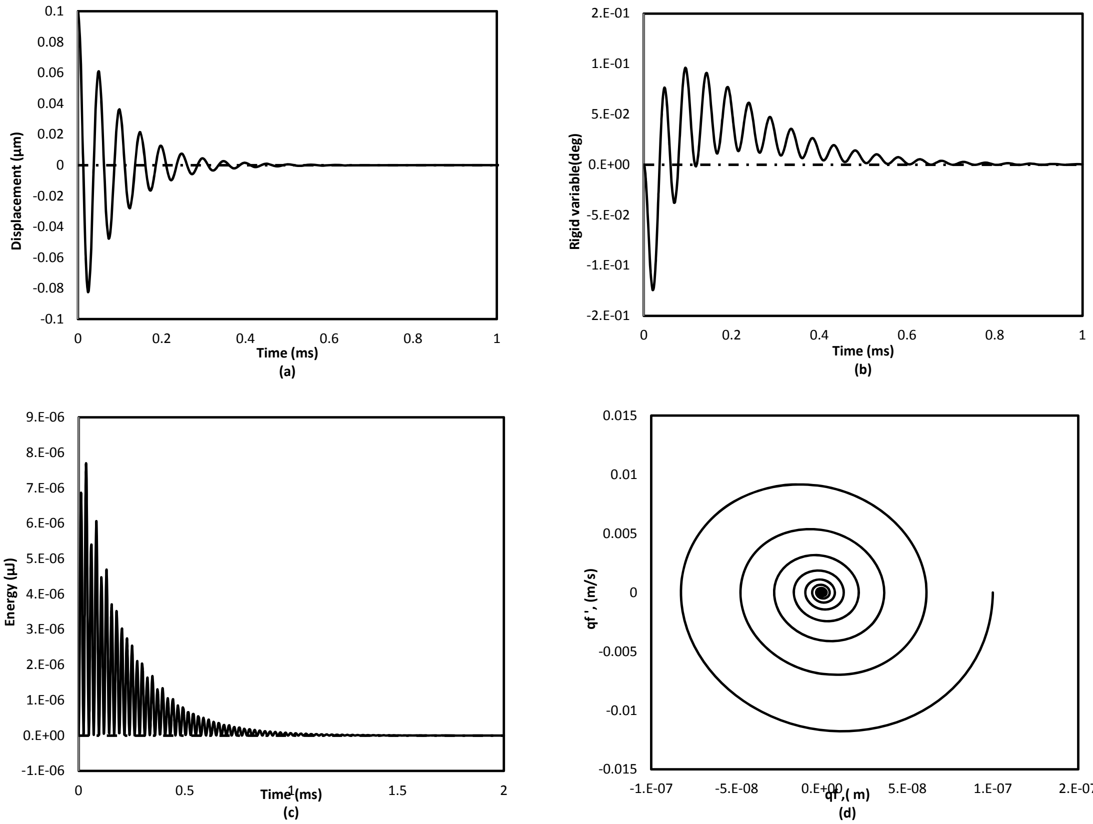
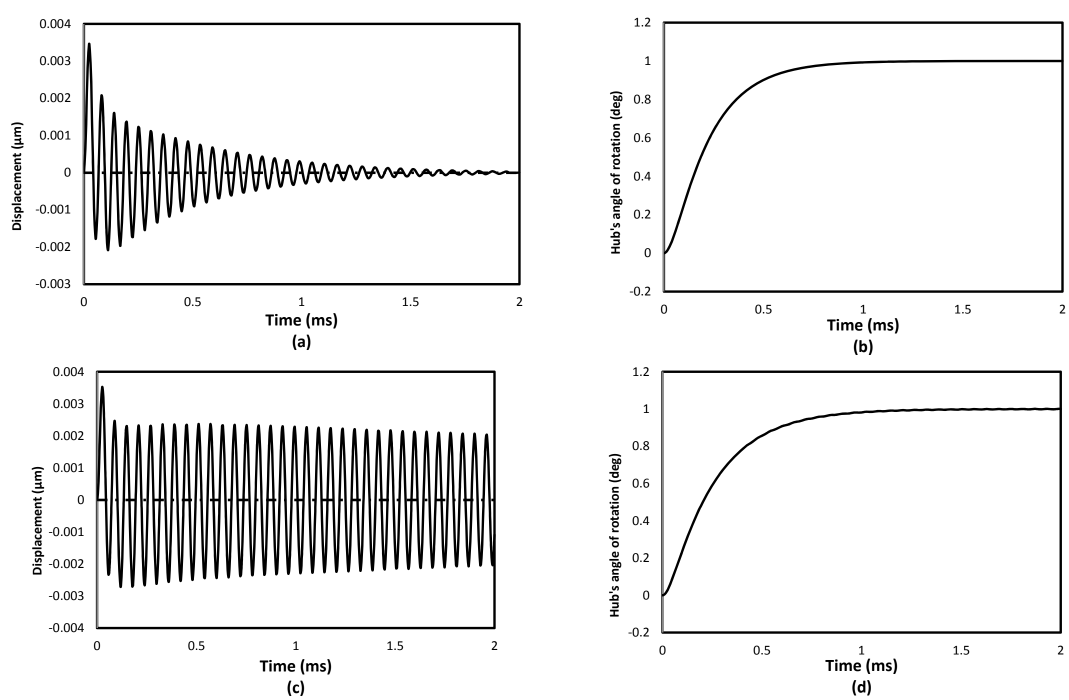


Figure 3. Simulation results for CL control with injected damping force(a) Time evolution of the beam displacement; (b) Time evolution of the hub's angle of rotation (rigid variable) ; (c) Total energy of closed-loop system; (d) plane phase diagram of flexible deflection.



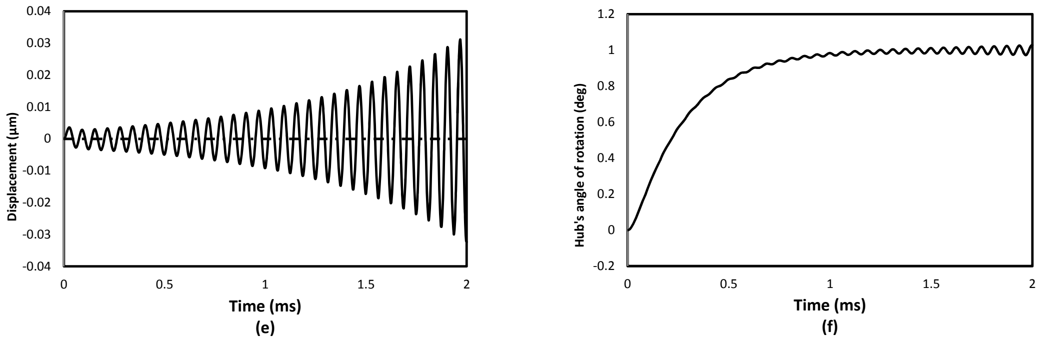


Figure 4- The effect of uncertainties on the stability of the system, asymptotically stability of (a) the beam displacement, (b) the rigid variable for $\tilde{l} = -0.06$; stability of (c) the beam displacement, (d) the rigid variable for $\tilde{l} = -0.07$; and instability of (e) the beam displacement, (f) the rigid variable for $\tilde{l} = -0.09$.

6 CONCLUSIONS

This work has presented a theoretical study of applying controlled Lagrangian method for tracking of a rotary microbeam. The potential of this technique in its application to under-actuated mechanical systems, in particular to a microbeam with rotating joint was investigated. For this goal, at first the nonlinear partial differential equations of motion of a rotary microbeam was derived employing Hamilton's principle. The Rayleigh–Ritz scheme was then applied to discretize the partial differential equations of motion into a set of second-order nonlinear ODEs. The controlled Lagrangian method is a control design method based on energy concepts, so the Lyapunov stability analysis of the closed-loop system that is obtained with the proposed control law studied. Stability and robustness stability of the control approach was analyzed, and the convergence of the tracking errors was proved theoretically. The simulated scheme, based on controlled Lagrangian method, has been shown to achieve acceptable regulation of the tip position of micro-cantilever beam. Also, undesirable vibration of the flexible variable was well damped.

REFERENCES

[1] s. M., R. M., K. M.H. and A. M.T., "On the size-dependent behavior of functionally graded micro-beams," *Materials and Design,* pp. 1435-1443, 2011.

[2] H. Y., D. G., S. X., G. D., C. B. and L. Z., "Preparing of a high speed bistable electromagnetic RF MEMS switch," *Sensors and Actuators A: Physical,* p. 532–537, 2007.

[3] S.-K. A., B. S. and J. N., "Modeling and experimental vibration analysis of nanomechanical cantilever active probes," *Journal of Micromechanics and Microengineering,* pp. 1-11, 2008.

[4] E. M., J. N. and D. M., "Dynamic modeling and performance evaluation of a vibrating microgyroscope under general support motion," *Journal of Sound and Vibration,* p. 146–164, 2007.

[5] G. M. H., F. H. and A. M., "Nonlinear behaviour of electrically actuated MEMS resonators," *International Journal of Engineering Science,* vol. 71, pp. 137-155, 2013.

[6] G. M. H., F. H. and A. M., "Nonlinear dynamics of a microscale beam based on the modified couple stress theory," *Composites: Part B,* pp. 318-324, 2013.

[7] R. S., "A micro scale geometrically non-linear Timoshenko beam model based on strain gradient elasticity theory," *International Journal of Non-Linear Mechanics,* vol. 47, p. 863–873, 2012.

[8] R. Vatankhah, A. Najafi, H. Salarieh and A. Alasty, "Boundary stabilization of non-classical micro-scale beams," *Applied Mathematical Modelling,* p. 8709–8724, 2013.

[9] M. Vagia, "A frequency independent approximation and a sliding mode control scheme for a system of a micro-cantilever beam," *ISA Transactions,* pp. 325-332, 2012.

[10] D. E. Chang, A. M. Bloch, N. E. Leonard, J. E. Marsden and C. A. Woolsey, "The equivalence of controlled Lagrangian and controlled Hamiltonian systems," *ESAIM: Control, Optimisation and Calculus of Variations,* vol. 8, pp. 393-422, 2002.

[11] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden and G. S. De Alvarez, "Stabilization of rigid body dynamics by internal and external torques," *Automatica,* vol. 28(4), p. 745–756, 1992.

[12] A. Sanz and . E. Victor, "Interconnection and damping assignment passivity-based experimental control of a single-link flexible robot arm," *IEEE International Conference on Control Applications,* pp. 2504-2509, 2006.

[13] A. Sanz and V. Etxebarria, "Experimental Control of Flexible Robot Manipulators," in *Robot Manipulators*, InTech, 2008.

[14] M. H. Ghayesh, M. Amabili and H. Farokhi, "Nonlinear forced vibrations of a microbeam based on the strain gradient elasticity theory," *International Journal of Engineering Science,* vol. 63, pp. 52-60, 2013.

[15] M. Kalyoncu, "Mathematical modelling and dynamic response of a multi-straight-line path tracing flexible robot manipulator with rotating-prismatic joint," *Applied Mathematical Modelling,* p. 1087–1098, 2008.

[16] S. S. Rao, Vibration of continuous systems, John Wiley & Sons., 2007.

[17] H. Baruh, ANALYTICAL DYNAMICS, WCB McGraw-Hill, 1999.

[18] M. H. Ghayesh, H. Farokhi and M. Amabili, "Coupled nonlinear size-dependent behaviour of microbeams," *Applied Physics A,* vol. 112(2), pp. 329-338, 2013.

[19] D. E. Chang, A. M. Bloch, N. E. Leonard, J. E. Marsden and C. A. Woolsey, "The equivalence of controlled lagrangian and controlled hamiltonian systems," *ESAIM: Control, Optimisation and Calculus of Variations,* pp. 393-422, 2002.

[20] R. Ortega, M. W. Spong, F. Gómez-Estern and G. Blankenstein, "Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment," *IEEE Transactions of Automatic Control,* p. 1218–1233, 2002.