8th GRACM International Congress on Computational Mechanics
Volos, 12 July – 15 July 2015

# EQUATIONS OF MOTION FOR MECHANICAL SYSTEMS SUBJECT TO EQUALITY MOTION CONSTRAINTS

**Sotirios Natsiavas, Elias Paraskevopoulos**

Department of Mechanical Engineering
Aristotle University
Thessaloniki, Greece
e-mail: natsiava@auth.gr

**Keywords:** Analytical Mechanics, Newton's law of motion, Motion constraints, Tangent and dual spaces, Lagrange multipliers.

**Abstract.** *A new set of equations of motion is presented for a class of mechanical systems subject to equality motion constraints. Specifically, the systems examined satisfy a set of holonomic and/or nonholonomic scleronomic constraints. The main idea is to consider the equations describing the action of the constraints as an integral part of the overall process leading to the equations of motion. The constraints are incorporated one by one, in a process analogous to that used for setting up the equations of motion. This proves to be equivalent to assigning appropriate inertia, damping and stiffness properties to each constraint equation and leads to a system of second order ordinary differential equations for both the coordinates and the Lagrange multipliers associated to the motion constraints automatically. This brings considerable advantages, avoiding problems related to systems of differential-algebraic equations or penalty formulations. Apart from its theoretical value, this set of equations is well-suited for developing new robust and accurate numerical methods.*

## 1 INTRODUCTION

Dynamics of mechanical systems subject to motion constraints is a research area which has received valuable contributions from many well known scientists and engineers, over a long period of time. One reason for this is the inherent theoretical beauty of this topic. Another reason is related to the large range of practical problems that can be solved by applying the results obtained in this scientific area. Since the pioneering work of Newton, Euler, Lagrange and Hamilton, a lot of research work has been carried over this subject up to now. This activity was especially intensive for problems in the areas of multibody dynamics and control.

In engineering applications, the spatial configuration of a system, involving a single or several bodies, is fully described by a set of Lagrangian coordinates [1]. For small order systems, it is easy and convenient to choose a minimum number of them. However, it is frequently beneficial to introduce more coordinates than those actually needed for describing the dynamics of a system. In such cases, extra equations must also be introduced, representing the effect of the constraints imposed on the motion [2, 3]. Frequently, the set of constraint equations is simply appended to another set of second order ordinary differential equations (ODEs), which arise by applying the laws of motion to all components of the system. This leads to a set of high index differential-algebraic equations (DAEs), which are associated with a singular mathematical set up, causing severe difficulties in trying to develop methods for their numerical solution [4]. A comprehensive review on these problems and methods developed to cure them can be found in [5]. However, the original source of the problem can be avoided by an appropriate incorporation of the constraints into the equations of motion.

The main objective of this work is to present a new set of equations of motion for a class of constrained mechanical systems in a systematic and consistent way. The emphasis is placed on dealing with discrete systems subject to scleronomic constraints. The approach developed is based on some fundamental concepts of differential geometry and treats both holonomic and nonholonomic constraints by considering them as part of the overall process of deriving the equations of motion. This leads to assignment of appropriate inertia, damping and stiffness properties to each of the constraints. As a result, the final set of equations of motion consists of second order ODEs [6]. This is accomplished in a quite natural manner and leads to elimination of singularities associated with DAE formulations. Moreover, there is no need to differentiate the constraint equations or introduce artificial parameters for scaling and stabilizing the system [5, 7].

The organization of this paper is as follows. First, some useful concepts of differential geometry are briefly summarized in the following section. This includes two conditions on the metric and connection of two manifolds, describing the motion of a system with a different number of constraints, so that the form of Newton's law of motion remains invariant [8]. This in turn sets up the ground for a consistent derivation of the equations of motion, which is a task completed in the following two sections. Then, a section devoted to a comparison of the new method with well-established methods in the field is included. Next, a selected example is presented, illustrating the straightforward application of the new method to mechanical systems. Finally, the most important conclusions are summarized in the last section.

## 2 MOTION OF A DYNAMICAL SYSTEM ON A CONFIGURATION MANIFOLD

The configuration of a mechanical system can frequently be determined by a finite number of generalized coordinates, $q^1,\ldots,q^n$. Then, the motion of the system can be represented by the motion of a point along a curve $\gamma=\gamma(t)$ on an $n$-dimensional manifold $M$. The tangent vector $\underline{v}=d\gamma/dt$ at a point $p$ belongs to an $n$-dimensional vector space $T_pM$, the tangent space at $p$. Therefore, if $\mathfrak{B}=\{\underline{e}_1 \quad \ldots \quad \underline{e}_n\}$ is a basis, any vector $\underline{u}$ of $T_pM$ can be expressed as $\underline{u}=\sum_{i=1}^n u^i\,\underline{e}_i = u^i\,\underline{e}_i$ by adopting the usual summation convention for repeated indices [9]. In addition, at every point $p$, one can define the dual or cotangent space $T_p^*M$. Then, for any vector $\underline{u}$, a covector $\underset{\sim}{u}^*$ may be identified through the dual product

$$\underset{\sim}{u}^*(\underline{w}) \equiv \langle \underline{u},\underline{w}\rangle, \quad \forall \underline{w}\in T_nM\,, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $T_pM$. In this way, to each basis $\{\underline{e}_i\}$ (with $i=1,\ldots,n$) of $T_pM$, a dual basis $\{\underset{\sim}{e}^i\}$ can be established for $T_p^*M$ by employing the condition

$$\underset{\sim}{e}^i(\underline{e}_i)=\delta_i^i\,, \tag{2}$$

where $\delta_j^i$ is a Kronecker's delta. Then, the covariant differential of a covector field $\underset{\sim}{u}^*(t)$ on $M$ along a vector $\underline{v}$ of $T_pM$ is evaluated in the form

$$\nabla_v \underset{\sim}{u}^*(t) = (\dot{u}_k - \Lambda_{ik}^j v^i u_i)\underset{\sim}{e}^k\,, \tag{3}$$

where $\Lambda_{jk}^i$ are the components of a connection $\nabla$ in the basis $\mathfrak{B}$ of $T_pM$, known as affinities [9, 10].

The true path of motion on a manifold is determined by application of Newton's second law. First, for a particle, it holds

$$\nabla_v \underset{\sim}{p}^* = \underset{\sim}{f}^*\,, \tag{4}$$

where covector $\underset{\sim}{p}^*$ stands for the generalized momentum of the particle, while $\underset{\sim}{f}^*$ represents the resultant external forcing on the particle [10]. Generalizing Eq. (4), the motion on a manifold $M$ is governed by Newton's law in the form

$$\nabla_v \underset{\sim}{p}_M^* = \underset{\sim}{f}_M^*\,, \tag{5}$$

with $\underset{\sim}{f}_M^* = f_i\underset{\sim}{e}^i$ [8]. Moreover, the generalized momentum is defined as the covector corresponding to the velocity vector, i.e., $\underset{\sim}{p}^* \equiv \underset{\sim}{v}^*$. Then, if $\underline{v}=v^i\underline{e}_i$ and $\underset{\sim}{p}^* = p_i\underset{\sim}{e}^i$, application of the definition Eq. (1) leads to

$$p_i = g_{ij}v^j\,, \tag{6}$$

where $g_{ij}=\langle \underline{e}_i,\underline{e}_j\rangle$ are components of the metric tensor at $p$. These quantities can be selected through the kinetic energy of the system [10]. Finally, it is useful to think of Newton's law (5) as a map, defined by the class of covectors

$$\underset{\sim}{h}_M^* \equiv \nabla_v \underset{\sim}{p}_M^* - \underset{\sim}{f}_M^*\,, \tag{7}$$

which will be mentioned as Newton covectors in the sequel.

Frequently, the motion of a system is subjected to an additional set of scleronomic constraints, cast in the form

$$A(q)\underline{v}=\underline{0}\,, \tag{8}$$

where $A=[a_i^R]$ is a $k\times n$ matrix and $\underline{v}$ belongs to $T_pM$. When all the constraints are holonomic, these equations can be integrated and put in the form

$$\underline{\phi}(q)=\underline{0}\,. \tag{9}$$

After imposing this additional set of constraints, the motion of the system takes place on a curve $\gamma_A(t)$ of another manifold $M_A$, with dimension $m=n-k$. Let $\{\theta^\alpha\}$ ($\alpha=1,\ldots,m$) be a set of coordinates in a neighborhood of a point $p_A$, related to point $p$ of $M$ through the mapping defined by the constraints. Furthermore, let $\{\underline{e}_\alpha\}$ be a basis of the tangent space $T_{p_A}M_A$ and $\{\underset{\sim}{e}^\alpha\}$ be the corresponding basis of the dual space $T^*_{p_A}M_A$. If the constraints are linearly independent and $\underline{v}_I \equiv \underline{\dot{\theta}}$ includes the independent components of the velocities, it easily turns out that

$$\underline{v} = N\underline{\dot{\theta}}, \tag{10}$$

where $N$ is an $n \times m$ matrix depending on the elements of $A$. This defines a linear transformation $E_S$, by

$$E_S = N^i_{\bullet\alpha}\underline{e}_i \otimes \underset{\sim}{e}^\alpha, \tag{11}$$

where symbol $\otimes$ represents the classical tensor product [9]. It also defines the dual operator

$$E_{SD} = N_\alpha^{\bullet i}\underset{\sim}{e}^\alpha \otimes \underline{e}_i. \tag{12}$$

Here and in the sequel, lower case Latin indices vary from 1 to $n$, while lower case Greek indices vary from 1 to $m$ .The mapping $E_{SD}$ helps in transferring the law of motion from $M$ to $M_A$. In particular, it is first required that the law governing the motion on the manifold $M_A$ preserves the form of Newton's law expressed by Eq. (5). This means that

$$\nabla_{v_A} p^*_A = f^*_A. \tag{13}$$

In analogy to Eq. (7), one can then define a class of Newton covectors on $T^*_{p_A}M_A$ with form $\underset{\sim}{h}^*_A \equiv \nabla_{\underline{v}_A}\underset{\sim}{p}^*_A - \underset{\sim}{f}^*_A$ and require that the following condition is also satisfied

$$\underset{\sim}{h}^*_A = E_{SD}\underset{\sim}{h}^*_M. \tag{14}$$

Based on these, it was shown in a previous study that if the motion on manifold $M$ is governed by Newton's law in the form Eq. (5), this form remains the same even on manifold $M_A$, provided that the following two conditions are satisfied

$$g_{\alpha\beta} = N^i_\alpha g_{ij} N^j_\beta \tag{15}$$

and

$$(\Lambda^\rho_{\gamma\alpha} g_{\rho\beta} - N^i_{\alpha,\gamma} N^j_\beta g_{ij} - N^i_\alpha N^j_\beta N^k_\gamma \Lambda^m_{ij} g_{mk}) v^\beta v^\gamma = 0, \tag{16}$$

where $\Lambda^\gamma_{\alpha\beta}$ and $g_{\alpha\beta}$ are the components of the connection and metric tensor at a point of $M_A$ [8].

## 3 OPERATORS ACTING BETWEEN TANGENT AND COTANGENT SPACES MANIFOLD

A key step in the process developed is that each of the $k$ equations of constraints is considered separately. Specifically, the constraints represented by Eq. (8) are viewed as dual products

$$\dot{\psi}^R(q, \underline{v}) \equiv (\underset{\sim}{a}^R(q))(\underline{v}) = 0, \qquad (R = 1, \ldots, k), \tag{17}$$

where the quantity $\underset{\sim}{a}^R(\underline{q})$ is the $R$-th row of matrix $A$ with elements $a^R_{\bullet i}$ and is treated as a covector of $T^*_pM$. The last relation provides the foundation to define the linear operator

$$T_R = a^R_{\bullet i}\underline{e}_R \otimes \underset{\sim}{e}^i, \tag{18}$$

representing a mapping from $T_pM$ to a single-dimensional space $W_R \equiv T_{p_R}M_R$, with basis vector $\underline{e}_R$. The underlying manifold $M_R$ and its point $p_R$ are also defined by the action of the $R$-th constraint. From hereon, an upper case Latin index varies from 1 to $k$ and does not obey the summation convention on repeated indices.

The operator $T_R$ is surjective. Therefore, it possesses a null space $H_R$ and accepts a right inverse $S_R$, which is an injective linear mapping from $W_R$ to a single-dimensional subspace $V_R$ of $T_pM$ [9]. In addition,

$$S_R = c^i_{\bullet R}\underline{e}_i \otimes \underset{\sim}{e}^R \quad \Rightarrow \quad S_R\underline{e}_R = c^i_R\underline{e}_i \equiv \underline{c}_R, \tag{19}$$

where vector $\underline{c}_R$ can be selected as basis of $V_R$, with components satisfying

$$c^i_R a^R_i = 1 \tag{20}$$

Finally, it is easy to show that the operator $\Pi_R \equiv S_R T_R$ is a projection, so that the space $T_pM$ can be split in the form

$$T_pM = H_R \oplus V_R, \tag{21}$$

where $H_R$ and $V_R$ are ($n-1$)- and one-dimensional vector subspaces of $T_pM$, with $R = 1, \ldots, k$, respectively.

Based on the projection theorem [9], this implies that any element $\underline{v}$ of $T_pM$ can be decomposed uniquely in the form

$$\underline{v} = \underline{u}_R + \underline{v}_R \,, \tag{22}$$

where $\underline{u}_R \in H_R$ and $\underline{v}_R \in V_R$. Moreover, the analysis can be used to define a global decomposition of $T_pM$ in the presence of multiple constraints. By successively applying all the constraints, the following overall splitting is obtained

$$T_pM = H_T \oplus V_S \,, \tag{23}$$

where $H_T$ is an ( $n-k$ )- and $V_S$ is a $k$ -dimensional vector subspace of $T_pM$ , defined by

$$H_T \equiv \bigcap_{R=1}^{k} H_R \quad \text{and} \quad V_S \equiv V_1 \oplus \cdots \oplus V_k \,, \tag{24}$$

respectively. In addition, one can create the $k$ -dimensional Cartesian product space

$$W \equiv W_1 \times \cdots \times W_k \cong \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}} \,,$$

which can be viewed as the tangent space of a $k$ -dimensional underlying manifold

$$M_C \equiv M_1 \times \cdots \times M_k \,. \tag{25}$$

Using the decomposition (23), an arbitrary vector of $T_pM$ can be expressed in the form

$$\underline{v} = \underline{u}_T + \underline{v}_S \,, \quad \text{with} \quad \underline{v}_S = \underline{v}_1 + \cdots + \underline{v}_k \,, \tag{26}$$

where $\underline{u}_T \in H_T$ and $\underline{v}_R \in V_R$ (see Fig. 1). The above can be used to define two composite transformations, $T : T_pM \to W$ and $S : W \to V_S$. In addition, a bijection $\hat{T}$ from $V_S$ to $W$ results by elimination of the null space $H_T$ of $T$ through a projection. To complete the picture for the tangent spaces, an injective operator $E_S$ is also defined, through Eq. (11), with a left inverse $\hat{E}_T$. A related operator $E_T$ from $T_pM$ to $T_{p_A}M_A$ can also be defined.

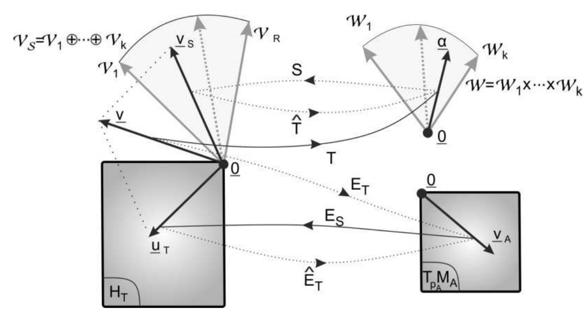


Figure 1. Transformations between the tangent spaces of manifolds $M$ , $M_A$ and $M_C$

An analogous picture is obtained for the cotangent spaces. In particular, two new operators are defined first according to

$$T_{RD} = a_i^{\cdot R} \underset{\sim}{e}^i \otimes \underline{e}_R \quad \Rightarrow \quad T_{RD} \underset{\sim}{e}^R = a_i^{\cdot R} \underset{\sim}{e}^i \equiv \underset{\sim}{c}^R \tag{27}$$

so that the covector $\underset{\sim}{c}^R$ provides a basis to the single-dimensional vector space $V_R^*$ and

$$S_{RD} = c_R^{\cdot i} \underset{\sim}{e}^R \otimes \underline{e}_i \quad \Rightarrow \quad S_{RD} \underset{\sim}{e}^i = c_R^{\cdot i} \underset{\sim}{e}^R \,. \tag{28}$$

Then, through the projection $\Pi_{RD} \equiv T_{RD} S_{RD}$ from $T_p^*M$ to $V_R^*$, the dual space $T_p^*M$ can be split in the form

$$T_p^*M = H_R^* \oplus V_R^* \,, \tag{29}$$

for each $R = 1, \ldots, k$ , where $H_R^*$ is an (n-1)-dimensional subspace of $T_p^*M$ . Then, for any element $\underset{\sim}{v}^*$ of $T_p^*M$

$$\underset{\sim}{v}^* = \underset{\sim}{u}^*_R + \underset{\sim}{v}^*_R \,, \tag{30}$$

where $\underset{\sim}{u}^*_R \in H_R^*$ and $\underset{\sim}{v}^*_R \in V_R^*$. Moreover, successive application of all the constraints leads to the splitting

$$T_p^*M = H_S^* \oplus V_T^* \,, \tag{31}$$

where $H_S^*$ is an ( $n-k$ )-dimensional and $V_T^*$ is a $k$ -dimensional subspace of $T_p^*M$ , defined by

$$H_S^* \equiv \bigcap_{R=1}^{k} H_R^* \quad \text{and} \quad V_T^* \equiv V_1^* \oplus \cdots \oplus V_k^* \,, \tag{32}$$

respectively. One can also construct the $k$ -dimensional Cartesian product space

$$W^* \equiv W_1^* \times \cdots \times W_k^* \cong \underbrace{\mathbb{R}^* \times \cdots \times \mathbb{R}^*}_{k \text{ times}} .$$

Based on the decomposition (31), any element of $T_p^* M$ can be expressed in the form $\underset{\sim}{v}^* = \underset{\sim}{u}_S^* + \underset{\sim}{v}_T^*$ with $\underset{\sim}{v}_T^* = \underset{\sim}{v}_1^* + \cdots + \underset{\sim}{v}_k^*$, where $\underset{\sim}{u}_S^* \in H_S^*$ and $\underset{\sim}{v}_R^* \in V_R^*$. This provides the ground to define two composite transformations, $S_D$ and $T_D$, between $T_p^* M$ and $W^*$ (see Fig. 2). The picture is completed by defining a bijection $\hat{S}_D$ from $V_T^*$ to $W^*$ and an operator $E_{SD}$ acting from $T_p^* M$ to $T_{p_A}^* M_A$. Finally, an operator $\hat{E}_{SD}$ is introduced, which is a bijection from $H_S^*$ to $T_{p_A}^* M_A$.

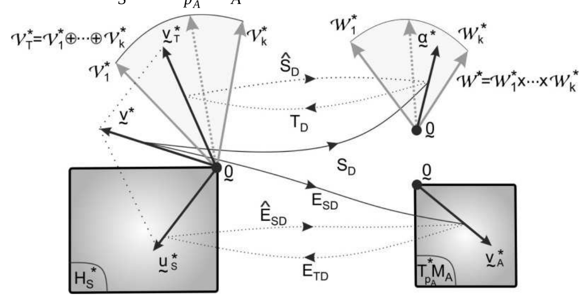


Figure 2. Transformations between the dual spaces of manifolds $M$ , $M_A$ and $M_C$

## 4 DERIVATION OF THE EQUATIONS OF MOTION ON THE ORIGINAL MANIFOLD

A central idea of the present work is based on satisfaction of the motion constraints on manifold $M$ by considering the equations of motion on $M_A$ and $M_C$ - or equivalently on $M_R$, with $R = 1, \ldots, k$ - as well. First, allow a potential violation of the $R$ -th constraint only and consider the class of Newton covectors

$$\underset{\sim}{h}_R^* = \dot{\underset{\sim}{p}}_R^* - \underset{\sim}{f}_R^* \tag{33}$$

on each of the dual spaces $W_R^*$. A typical element of $W_R^*$ can be put in the form

$$\underset{\sim}{\alpha}_R^* = \alpha_R \underset{\sim}{e}^R = g_{RR} \dot{q}^R \underset{\sim}{e}^R , \tag{34}$$

where $q^R$ is the coordinate of a point $p_R$ of the space $M_R$, as shown in Fig. 3. Moreover, based on the compatibility conditions (15) and (16), the metric and connection on $M_R$ are selected by

$$g_{RR} = c_R^i g_{ij} c_R^j \quad \text{and} \quad \Lambda_{RR}^R = 0 . \tag{35}$$

Therefore, taking into account Eqs. (6) and (34), the first term of $\underset{\sim}{h}_R^*$ in Eq. (33) can be put in the form

$$\dot{\underset{\sim}{p}}_R^* = (g_{RR} \dot{q}^R)^{\boldsymbol{\cdot}} \underset{\sim}{e}^R . \tag{36}$$

Next, by applying Eq. (28), the second term of $\underset{\sim}{h}_R^*$ is obtained through the transformation

$$\underset{\sim}{f}_R^* = S_{RD} \underset{\sim}{f}_M^* = \overline{f}_R \underset{\sim}{e}^R . \tag{37}$$

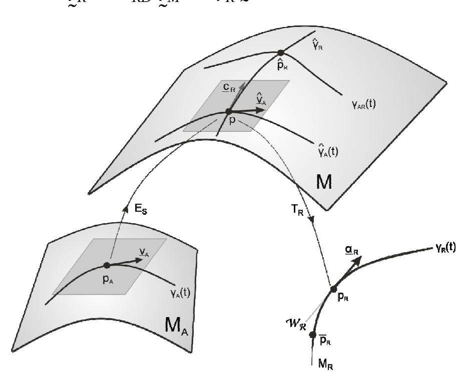


Figure 3. Solution path and deviation along direction of $R$ -th constraint on manifold $M$

The term $\overline{f}_R = c_R^i f_i$ is the component of the force developed when the $R$-th constraint, represented by Eq. (17), tends to be violated. In general, $f_i = f_i(\underline{q}, \dot{\underline{q}}, t)$ [3, 10]. Then, its value at a neighboring point $\hat{p}_R$ is

$$\hat{f}_i = f_i(\underline{q} + s^R \underline{c}_R, \dot{\underline{q}} + \dot{s}^R \underline{c}_R, t) .$$

The quantitiy $s^R$ is the canonical coordinate along the autoparallel of $M$ starting at $p$ with tangent $\underline{c}_R$. Then,

$$\hat{f}_i = f_i(q, \dot{\underline{q}}, t) + \left[ \frac{\partial f_i}{\partial q}(q, \dot{\underline{q}}, t)\underline{c}_R \right] s^R + \left[ \frac{\partial f_i}{\partial \dot{\underline{q}}}(q, \dot{\underline{q}}, t)\underline{c}_R \right] \dot{s}^R + \ldots \tag{38}$$

ince the values of $s^R$ and $\dot{s}^R$ are infinitesimal. Next, the choice $s^R = q^R$ is made. Combination of Eqs. (36)-(38) and substitution in Eq. (33) yields the following expression for the Newton covectors on $M_R$

$$\underset{\sim}{h}_R^* = h_R \underset{\sim}{e}^R = [(\overline{m}_{RR}\dot{q}^R)^{\boldsymbol{\cdot}} + \overline{c}_{RR}\dot{q}^R + \overline{k}_{RR} q^R - \overline{f}_R] \underset{\sim}{e}^R , \tag{39}$$

where the new coefficients in the above expression are defined by

$$\overline{m}_{RR} = g_{RR}\,,\;\; \overline{c}_{RR} = -c_R^i \frac{\partial f_i}{\partial \dot{q}^j}(q, \dot{\underline{q}}, t) c_R^j \;, \text{-} \overline{k}_{RR} = -c_R^i \frac{\partial f_i}{\partial q^j}(q, \dot{\underline{q}}, t) c_R^j \,,\;\; \overline{f}_R = c_R^i f_i \tag{40}$$

Now, the Newton covectors on manifolds $M$ and $M_R$ are related by

$$\underset{\sim}{h}_R^* = S_{RD} \underset{\sim}{h}_M^* . \tag{41}$$

Next, multiplying both sides of the last equation from the left by $T_{RD}$, projects these equations back to $T_p^* M$ and more specifically to $V_R^*$. In particular, this leads to

$$\underset{\sim}{v}_R^* = T_{RD} \underset{\sim}{h}_R^* = \Pi_{RD} \underset{\sim}{h}_M^* . \tag{42}$$

Likewise, multiplying both sides of Eq. (15) from the left by $E_{TD}$, projects these equations back to $T_p^* M$ and more specifically to $H_S^*$ (see Fig. 2) by

$$\underset{\sim}{u}_S^* = E_{TD} \underset{\sim}{h}_A^* = \Pi_{ED} \underset{\sim}{h}_M^* . \tag{43}$$

Combining the above and using the decomposition (23) leads to

$$\underset{\sim}{h}_M^* = \underset{\sim}{h}_S^* + \underset{\sim}{h}_T^* = \Pi_{ED} \underset{\sim}{h}_M^* + \Pi_{TD} \underset{\sim}{h}_M^* , \tag{44}$$

or equivalently

$$\underset{\sim}{h}_M^* = E_{TD} \underset{\sim}{h}_A^* + \sum\nolimits_{R=1}^{k} T_{RD} \underset{\sim}{h}_R^* . \tag{45}$$

However, the equations of motion must always be satisfied on manifold $M_A$. This means that

$$\underset{\sim}{h}_M^* = \sum\nolimits_{R=1}^{k} T_{RD} \underset{\sim}{h}_R^* . \tag{46}$$

Moreover, direct application of Eq. (27) gives

$$T_{RD} \underset{\sim}{h}_R^* = T_{RD}(h_R \underset{\sim}{e}^R) = a_i^R h_R \underset{\sim}{e}^i . \tag{47}$$

Finally, the coordinates $q^R \equiv \lambda^R$ can be viewed as Lagrange multipliers and the equations of motion (46) take the form

$$(g_{ij}\dot{q}^j)^{\boldsymbol{\cdot}} - \Lambda_{ki}^m g_{mj}\dot{q}^k \dot{q}^j \quad = f_i + \sum\nolimits_{R=1}^{k} a_i^R [(\overline{m}_{RR}\dot{\lambda}^R)^{\boldsymbol{\cdot}} + \overline{c}_{RR}\dot{\lambda}^R + \overline{k}_{RR}\lambda^R - \overline{f}_R] \tag{48}$$

For each holonomic constraint, coordinate $s^R$ coincides with the function of constraint $\phi^R(\underline{q})$. Through the choice $q^R = s^R$, this defines a map from $M$ to $M_R$ with explicit form

$$q^R(t) = \phi^R(q(t)) , \tag{49}$$

where $\underline{q}(t)$ represents the coordinates of point $p(t)$ on $M$. Then, the tangent map from $T_p M$ to $T_{p_R} M_R$ is

$$\dot{q}^R(t) = \dot{\phi}^R(q(t)) , \tag{50}$$

which specifies the corresponding velocity coordinate as well. Therefore, when the $R$-th constraint is satisfied, then taking into account Eq. (9), relations Eqs. (49) and (50) yield $q^R(t) = 0$ and $\dot{q}^R(t) = 0$ ,respectively, for all times. This implies immediately that $\ddot{q}^R(t) = 0$

Consequently, based on Eq. (39), the class of Newton covectors on $W_R^*$, satisfying the motion constraints, is represented by

$$\underset{\sim}{h}_R^N = -\overline{f}_R \underset{\sim}{e}^R \,. \tag{51}$$

When the $R$ -th constraint is not satisfied, a scalar equation with form

$$(\overline{m}_{RR}\dot{\phi}^R)^{\boldsymbol{\cdot}} + \overline{c}_{RR}\dot{\phi}^R + \overline{k}_{RR}\phi^R = 0 \tag{52}$$

is obtained for each holonomic constraint, eventually, which forces both $\dot{\phi}^R$ and $\phi^R$ to become zero. Finally, when the constraint is nonholonomic, only the following tangent mapping is available $\dot{q}^R(t) = \dot{\psi}^R(q(t), \underline{\dot{q}}(t))$ Consequently, by following a similar path, it eventually leads to a scalar equation

$$(\overline{m}_{RR}\dot{\psi}^R)^{\boldsymbol{\cdot}} + \overline{c}_{RR}\dot{\psi}^R = 0 \,, \tag{53}$$

which can force only $\dot{\psi}^R$ to become zero.

Conditions (52) and (53) furnish a set of $k$ second order ODEs, which together with Eq. (48) provide a set of $n+k$ second order ODEs in the $n+k$ unknowns $q^i$ and $\lambda^R$. Note that conditions Eqs. (52) and (53) have a similarity to but are more general than those employed in the so-called Baumgarte stabilization [5, 7]. Here, these equations were derived as part of the systematic approach developed and were not introduced artificially. Also, all the coefficients of these equations were determined analytically and not through an adhoc selection.

## 5 COMPARISON WITH EXISTING METHODS

Besides its inherent beauty, the method presented provides a concrete way for treating the equations of motion of constrained systems in an accurate and efficient manner. For instance, the present approach brings a major theoretical advantage when compared to other approaches applied so far in the field of Analytical Dynamics and Multibody Dynamics in particular. This is related to a physically consistent elimination of the singularities associated with the sets of DAEs of motion. Here, the introduction of the Lagrange multipliers associated to the motion constraints is based on a dynamic treatment of the constraint equations, which is consistent with that of the main equations of motion. As a result, the derivatives of the Lagrange multipliers appear naturally in these equations and there is no reason to perform extra differentiations of the constraint equations. In addition, there is no need for an arbitrary and externally imposed numerical stabilization [5, 7]. Furthermore, all the constraints are introduced automatically and possess a proper numerical scaling, in contrast to penalty formulations, which are based on an adhoc introduction of terms and selection of parameters [11].

To make a comparison with existing formulations in a more explicit manner, matrix notation is employed next, with

$$\underline{q} = (q^1 \quad \cdots \quad q^n)^T \,, \quad \underline{\lambda} = (\lambda^1 \quad \cdots \quad \lambda^k)^T \,, \quad M = [g_{ij}] \text{ and } \underline{f}(t) = (f_1 \quad \cdots \quad f_n)^T$$

Then, the set of equations (48) can be put in the general form

$$\begin{gathered} M(\underline{q})\underline{\ddot{q}} = \underline{g}(\underline{q}, \underline{\dot{q}}, t) + A^T(\underline{q})[(\overline{M}\underline{\dot{\lambda}})^{\boldsymbol{\cdot}} + \overline{C}\underline{\dot{\lambda}} + \overline{K}\underline{\lambda} - \underline{\overline{f}}\,], \\ \underline{g}(\underline{q}, \underline{\dot{q}}, t) = \underline{f}(t) - \underline{h}(\underline{q}, \underline{\dot{q}}, t) - \dot{M}(\underline{q})\underline{\dot{q}} \end{gathered} \tag{54}$$

The term $\underline{h}(\underline{q}, \underline{\dot{q}})$ arises in the presence of nonzero affinities on the original manifold $M$ and includes the classical quadratic velocity terms [12], while the elements of the diagonal matrices

$$\overline{M} = diag(\overline{m}_{11} \quad \cdots \quad \overline{m}_{kk}) \,, \quad \overline{C} = diag(\overline{c}_{11} \quad \cdots \quad \overline{c}_{kk}) \,, \overline{K} = diag(\overline{k}_{11} \quad \cdots \quad \overline{k}_{kk})$$

are determined by Eq. (40).

The major difference with the classical approaches lies in the last term of Eq. (54), representing the constraint forces. Specifically, in all current analytical formulations, only the “static” term $A^T\underline{\lambda}$ appears in its place, so that the equations of motion are cast in the form

$$M(\underline{q})\underline{\ddot{q}} = \underline{g}(\underline{q}, \underline{\dot{q}}, t) + A^T(\underline{q})\underline{\lambda} \,. \tag{55}$$

In addition, the new approach takes the constraints from their original form (8) and puts them in the form (52) or (53), for each holonomic or nonholonomic constraint, respectively.

The equations of motion (55) are known as Lagrange’s equations of the first kind [5]. Usually, these equations appear in a highly sparse form, which is a desirable feature in numerical codes. However, together with the equation of holonomic constraints Eq. (9), they constitute a set of index-3 DAEs, which is known to present severe scaling problems [4]. These problems may be alleviated by reducing the index of the DAE to 2 or 1, through differentiation of the constraint equations Eqs. (8) or (9). Unfortunately, this brings other problems into the picture, related to constraint violation [5]. An alternative way to avoid problems associated to the DAE nature of Eq. (55) is based on developing methods transforming the equations of motion to an ODE form, based on an elimination of the Lagrange multipliers. A comprehensive classification and comparison of such methods is presented in [5]. Based on that, a brief account of some of those methods and their comparison with the method developed is given next. First, methods based on a minimal set of coordinates are examined, followed by investigation of methods based on redundant coordinates.

In previous work, one way to avoid DAE related problems was based on techniques of coordinate partitioning [13], which are equivalent to examining the motion on manifold $M_A$. As an alternative, another method was also developed, based on appropriate velocity transformations [14]. For open loop systems this method leads directly to a minimal set of coordinates. For closed loop systems, the equations of motion are first derived on a reduced set of coordinates. These equations appear in a DAE and not in an ODE form. In a second step, a further transformation to minimal coordinates can be performed, bringing the equations of motion to an ODE form on the base manifold $M_A$.

In general, the class of methods employing a minimal set of coordinates is based on creating a velocity relation similar to Eq. (10). This, in conjunction with Eq. (8), yields

$$AN = 0\,, \tag{56}$$

showing that the columns of matrix $N$ provide a basis for the null space of the constraint matrix $A$. Then, the last equation in combination with Eq. (55) leads easily to

$$\hat{M}\,\ddot{\underline{\theta}} = \underline{\hat{g}}\,, \tag{57}$$

With

$$\hat{M} = N^T M\, N \quad \text{and} \quad \underline{\hat{g}} = N^T(\underline{g} - M\dot{N}\dot{\underline{\theta}})\,, \tag{58}$$

representing a set of $m = n - k$ second order ODEs, requiring no scaling. These methods involve a minimum number of coordinates, but this is inconvenient since it is only locally valid and requires frequent update of the independent set of coordinates and the equations of motion. It is also related to loss of sparseness of the equations, due to execution of the operations indicated by Eq. (58). Moreover, the evaluation of the velocity transformation matrix and its time derivative must be performed at any time, while the constraints are satisfied up to the velocity level only. Finally, these methods run into singularities when $A$ is not a full rank matrix.

As an alternative, another class of methods leading to ODEs was also developed, keeping the coordinates of the original manifold $M$. A well known representative of this class is the index-1 formulation, based on a time differentiation of the constraint equation, leading to

$$A\underline{\ddot{q}} = \dot{A}\underline{\dot{q}} \equiv \underline{c}\,. \tag{59}$$

Then, proper manipulation of Eq. (55), with a simultaneous application of Eqs. (56) and (59) yields first the Lagrange multipliers in the explicit form

$$\underline{\lambda} = -(AM^{-1}A^T)^{-1}(\underline{c} - AM^{-1}\underline{g})\,. \tag{60}$$

Therefore, by a substitution of the last equation in Eq. (55), the following result is obtained

$$M\,\underline{\ddot{q}} = \underline{g} + A^T(AM^{-1}A^T)^{-1}(\underline{c} - AM^{-1}\underline{g})\,, \tag{61}$$

which is a set of $n$ second order ODEs in $\underline{q}$, requiring no scaling also. However, as indicated by Eq. (60), a full rank constraint matrix $A$ is required. Moreover, this formulation enforces constraints at the acceleration level, which causes more severe constraint violation problems than the previous class of methods.

A quite similar method is the so called null space formulation [5]. According to this method, proper manipulation of Eq. (55), with a simultaneous application of Eqs. (56) and (59) yields directly the following system of $n$ second order ODEs in the original generalized set of coordinates $\underline{q}$

$$\begin{bmatrix} N^T M \\ A \end{bmatrix}\underline{\ddot{q}} = \begin{pmatrix} N^T\underline{g} \\ \underline{c} \end{pmatrix}.$$

Again, this formulation enforces constraints at the acceleration level. Moreover, it requires scaling of the constraint equations and a full rank constraint matrix $A$. Better analytical performance is achieved by application of the Udwadia and Kalaba's formulation, which leads to the following system of $n$ second order ODEs for the original coordinates $\underline{q}$

$$M\,\underline{\ddot{q}} = \underline{g} + M^{1/2}(AM^{-1/2})^{+}(\underline{c} - AM^{-1}\underline{g})\,. \tag{62}$$

This method satisfies the constraints at the acceleration level also but, through the use of the Moore-Penrose generalized inverse matrix appearing in Eq. (62), it does not need scaling and can handle cases involving redundant constraints [3].

Finally, other well known methods leading to ODEs present similar features [5, 15, 16]. The good performance of the present formulation is due to the new terms in Eqs. (52)-(54). First, the equations of constraint require no stabilization, since Eq. (52) forces both $\dot{\phi}^R$ and $\phi^R$ to zero, while Eq. (53) forces $\dot{\psi}^R$ to zero for each holonomic and nonholonomic constraint, respectively. In addition, the terms $\bar{m}_{RR}$ cause a natural scaling of the constraint equations (52) and (53). These terms have a similar advantageous scaling effect even in the equations of motion (54). This effect is quite beneficial since the Lagrange multipliers $\underline{\lambda}$ in Eq. (54) represent coordinates (in the constraint manifold $M_C$), in contrast to the Lagrange multipliers in Eq. (55), representing forces [5]. Furthermore, with a small modification, affecting the dimension of the horizontal and vertical subspaces of the tangent spaces, the new method works even for rank deficient constraint matrices. Moreover, besides the formulation of the mass matrix $M$, the constraint matrix $A$ and the forcing array $\underline{g}$, which are common in all formulations, the present formulation updates only diagonal matrices and preserves the structure (and sparseness) of the original equations.

The previous ODE formulations make the geometry of the original manifold dependent on the nature of the additional constraints. In the present method, the properties of $M$ are kept independent on these constraints. More importantly, the previous methods cause an elimination of terms in the equation of motion, which are associated with the mechanism of correcting a potential constraint violation. Specifically, these are the terms in Eq. (54) not appearing in Eq. (55). On the other hand, the extra requirements for application of the present methods to mechanical problems are minimal, as illustrated by an example in the following section.

A major reason for the advantages of the present method is the full exploration of the geometrical properties of suitable operators, acting between the tangent and dual spaces at each point of abstract manifolds where the motion is viewed, in a complementary manner. In addition, the present method examines the effect of each constraint independently, leading to significant simplifications in the analysis. It also extends and generalizes the theoretical meaning of the Lagrange multipliers, describing the constraint forces, by assigning an equivalent inertia to them. This in turn leads to a set of equations of motion represented fully by ODEs. All these advantages provide a solid ground in developing more accurate and effective new discretization schemes for the numerical integration of the equations of motion of constrained systems [5, 7].

## 6 AN EXAMPLE

The analytical results presented are complemented next by an example, shown in Fig. 4. This system is subjected to known external forces and supported by linear springs and dampers. When no interconnection exists, the motion of this system is described by two coordinates, $q^1$ and $q^2$. Therefore, the configuration manifold is $M \equiv \mathbb{R}^2$ (with dimension $n = 2$), while the kinetic energy of the system can be put in the form

$$T = \tfrac{1}{2} m_1 (v^1)^2 + \tfrac{1}{2} m_2 (v^2)^2 ,$$

with $v^i = \dot{q}^i$ ($i = 1, 2$). Then, the components of the metric tensor are given by the matrix

$$G = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{63}$$

Moreover, since $M$ is in fact the product space $\mathbb{R} \times \mathbb{R}$, the connection on $M$ has components

$$\Lambda^k_{ij} = 0 . \tag{64}$$

Finally, the forces exerted on the particles appear in the form

$$f_i(\underline{q}, \underline{v}, t) = f_i(t) - k_{ij} q^j - c_{ij} v^j , \tag{65}$$

where $k_{ij}$ and $c_{ij}$ are elements of diagonal $2 \times 2$ matrices. Therefore, direct substitution in Eq. (5) yields the equations of motion of the free system on manifold $M$ in the form

$$m_1 \dot{v}^1 + c_1 v^1 + k_1 q^1 = f_1(t) , \quad m_2 \dot{v}^2 + c_2 v^2 + k_2 q^2 = f_2(t) \tag{66}$$

which is the same with that obtained by direct application of Newton’s law to each particle.

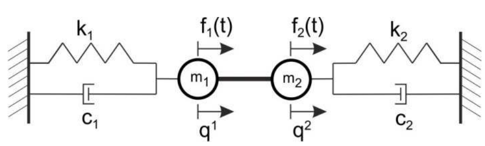


Figure 4. A two mass constrained system

Next, the motion of the particles is restricted by a single holonomic constraint ($k = 1$), through a rigid interconnection element shown in Fig. 4. This constraint appears in the form

$$\phi^1(q^1,q^2) = q^1 - q^2 = 0\,. \tag{67}$$

Therefore, the dimension of the corresponding manifold $M_A$, where the constrained motion takes place, is $m=1$. After differentiation, the constraint equation (67) takes the form

$$\dot{\psi}^1 = \dot{\phi}^1 = \dot{q}^1 - \dot{q}^2 = 0\,, \tag{68}$$

which is similar to Eq. (8), with an $1\times 2$ constraint matrix $A(q^1,q^2)=[1 \quad -1]$. Then, in accordance to Eqs. (20) and by substituting Eqs. (63) and (65) into Eq. (40) leads to

$$\bar{m}_{11} = \tfrac{1}{4}(m_1+m_2)\,,\quad \bar{c}_{11} = \tfrac{1}{4}(c_1+c_2)\,,\quad \bar{k}_{11} = \tfrac{1}{4}(k_1+k_2)\,,\quad \bar{f}_1 = \tfrac{1}{2}(f_1 - f_2)\,. \tag{69}$$

Then, based on Eqs. (48) and (52), the equations of motion of the constrained system on $M$ are cast in the form

$$(m_1 v^1)^{\bullet} + c_1 v^1 + k_1 q^1 - [(\bar{m}_{11}\dot{\lambda}^1)^{\bullet} + \bar{c}_{11}\dot{\lambda}^1 + \bar{k}_{11}\lambda^1 - \bar{f}_1] = f_1(t)\,, \tag{70}$$

$$(m_2 v^2)^{\bullet} + c_2 v^2 + k_2 q^2 + [(\bar{m}_{11}\dot{\lambda}^1)^{\bullet} + \bar{c}_{11}\dot{\lambda}^1 + \bar{k}_{11}\lambda^1 - \bar{f}_1] = f_2(t)\,, \tag{71}$$

Where $\lambda^1$ is the dynamic Lagrange multiplier corresponding to the constraint defined by Eq. (67) and

$$(\bar{m}_{11}\dot{\phi}^1)^{\bullet} + \bar{c}_{11}\dot{\phi}^1 + \bar{k}_{11}\phi^1 = 0\,. \tag{72}$$

## 7 SYNOPSIS

A new set of equations of motion for mechanical systems subject to scleronomic motion constraints was presented. Both holonomic and nonholonomic equality constraints were treated concurrently. The underlying idea was to incorporate the constraints one by one, in a process analogous to that used for setting up the equations of motion, which was equivalent to assigning appropriate inertia and possibly damping and stiffness properties to each constraint equation. This led to a system of second order ODEs for both the coordinates and a set of dynamic Lagrange multipliers, which eliminated the singularities associated with sets of DAEs and the need for an arbitrary selection of terms and parameters required by penalty formulations. Apart from its theoretical elegance, this is expected to have a significant impact in developing efficient methods for the numerical integration of these equations, since no stabilization or scaling is required. In addition, the new equations include terms activating a mechanism for performing constraint recovery in an automatic manner.

## 8 REFERENCES


[1] Greenwood, D.T., 1988, *Principles of Dynamics.* Prentice-Hall Inc., Englewood Cliffs, New Jersey.

[2] Neimark, J.I. and Fufaev, N.A., 1972, Dynamics of Nonholonomic Systems, *Translations of Mathematical Monographs, American Mathematical Society*, 33.

[3] Udwadia, F.E. and Kalaba, R.E., 1996, A*nalytical Dynamics A New Approach,* Cambridge University Press, Cambridge, UK.

[4] Brenan, K.E., Campbell, S.L. and Petzhold, L.R., 1989, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations,* North-Holland, New York.

[5] Bauchau, O.A., 2011, *Flexible Multibody Dynamics*, Springer Science+Business Media B.V., London.

[6] Natsiavas, S. and Paraskevopoulos, E., 2015, A Set of Ordinary Differential Equations of Motion for Constrained Mechanical Systems, *Nonlinear Dyn.* (DOI: 10.1007/s11071-014-1783-5).

[7] Geradin, M. and Cardona, A., 2001, *Flexible Multibody Dynamics*, John Wiley & Sons, New York.

[8] Paraskevopoulos, E. and Natsiavas, S., 2013, On Application of Newton's Law to Mechanical Systems with Motion Constraints, *Nonlinear Dyn.*, 72, pp. 455-475.

[9] Bowen, R.M. and Wang, C.-C., 2008, *Introduction to Vectors and Tensors, second ed.,* Dover Publications, Inc., New York.

[10] Bloch, A.M., 2003, *Nonholonomic Mechanics and Control,* Springer-Verlag New York Inc., New York.

[11] Bayo, E. and Ledesma, R., 1996, Augmented Lagrangian and Mass-Orthogonal Projection Methods for Constrained Multibody Dynamics, *Nonlinear Dyn.,* 9, pp. 113-130.

[12] Shabana, A.A., 2005, *Dynamics of Multibody Systems, third ed.* Cambridge University Press, New York.

[13] Wehage, R.A. and Haug, E.J., 1982, Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems, *ASME J. Mech. Des.* 104, pp. 247-255.

[14] Nikravesh, P.E. and Gim, G., 1993, Systematic Construction of the Equations of Motion for Multibody Systems Containing Closed Kinematic Loops, *ASME J. Mech. Des.,* 115, pp. 143-149.

[15] Essen, H., 1994, On the Geometry of Nonholonomic Dynamics, *ASME J. Appl. Mech.,* 61, pp. 689-694.

[16] Papastavridis, J.G., 1999, *Tensor Calculus and Analytical Dynamics*, CRC Press, Boca Raton.